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*Research Article*

# Convergence Rate of Some Two-Step Iterative Schemes in Banach Spaces

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This article proves some theorems to approximate fixed point of Zamfirescu operators on normed spaces for some two-step iterative schemes, namely, Picard-Mann iteration, Ishikawa iteration, S-iteration, and Thianwan iteration, with their errors. We compare the aforementioned iterations using numerical approach; the results show that S-iteration converges faster than other iterations followed by Picard-Mann iteration, while Ishikawa iteration is the least in terms of convergence rate. These results also suggest the best among two-step iterative fixed point schemes in the literature.

## 1. Introduction

Fixed point theory takes a large amount of literature, since it provides useful tools to solve many problems that have applications in different fields like engineering, economics, chemistry, game theory, and so forth. However, to find fixed points is not an easy task; that is why we use iterative methods for computing them. By time, many iterative methods have been developed and it is impossible to cover them all.

In the last four decades, numerous papers were published on the iterative approximation of fixed points of self- and non-self-contractive type operators in metric spaces, Hilbert spaces, or several classes of Banach spaces, while, for strict contractive type operators, the Picard iteration can be used to approximate the unique fixed point, for operators satisfying slightly weaker contractive type conditions, instead of Picard iteration, which does not generally converge; it was necessary to consider other fixed point iteration procedures. The Krasnoselskij iteration, the Mann iteration, and the Ishikawa iteration are certainly the most studied of these fixed point iteration procedures. Other iterations which have been studied also are Implicit Mann, Implicit Ishikawa, Thianwan, S-iteration, and hybrid Picard-Mann iterations. Recently, Wahab and Rauf [1] obtained some results on a faster implicit hybrid Kirk-multistep schemes for contractive type operators, to mention but a few.

Our aim in this paper is to establish the convergence and convergence rate of some two-step iterative schemes with errors using Zamfirescu operator in Banach spaces.

## 2. Preliminaries

The Picard iteration process is defined by the sequence

$$x_{n+1} = Tx_n, \quad n \geq 0, \tag{1}$$

and the concept of Picard iteration process with error is defined as follows:

$$x_{n+1} = Tx_n + u_n, \quad n \geq 0, \tag{2}$$

where $\{u_n\}$ satisfy $\sum_{n=1}^{\infty} \|u_n\| < \infty$.

In [2], Banach proved the convergence of (1) with the aid of the following contractive mapping:

$$d(Tx, Ty) \leq ad(x, y), \tag{3}$$

where $T : X \to X$, $(X, d)$ is a metric space, $a \in (0, 1)$, and $x, y \in X$. Banach's theorem is given as follows.

**Theorem 1.** *Let $(X, d)$ be a metric space and $T : X \to X$ be a contraction map on $X$. Then $T$ has a unique fixed point $x \in X$.*

When condition (3) is weaker, Picard iteration (1) will no longer converge to a fixed point. So, other iterations such as Mann iteration, Picard-Mann iteration, Ishikawa iteration, S-iteration, and Thianwan iteration would be considered.

Mann [3] defined an iteration process by the sequence, for $x_0 \in C$, $C$ of a closed subset of normed space:

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T x_n, \quad n \geq 0, \tag{4}$$

where $\alpha_n$ is a sequence in $[0, 1]$.

The concept of Mann iteration with error was discussed in [4] and it is given by the sequence

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T x_n + u_n, \quad n \geq 0, \tag{5}$$

where $\alpha_n$ is a sequence in $[0, 1]$ and $u_n$ satisfy $\sum_{n=1}^{\infty} \|u_n\| < \infty$.

Ishikawa [5] defined another iteration process given by the sequence

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, y_n + \alpha_n T x_n, \\ y_n &= (1-\beta_n)\, x_n + \beta_n T x_n, \ n \geq 0, \end{aligned} \tag{6}$$

where $\alpha_n$ and $\beta_n$ are sequences in $[0, 1]$. The Ishikawa iteration is a double Mann iteration and has better approximation than Mann iteration (5). The Ishikawa iteration with errors is given as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, y_n + \alpha_n T x_n + e_n, \\ y_n &= (1-\beta_n)\, x_n + \beta_n T x_n + f_n, \ n \geq 0, \end{aligned} \tag{7}$$

where $e_n$ and $f_n$ satisfy $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\sum_{n=1}^{\infty} \|f_n\| < \infty$. See [4].

In an attempt to reduce computational cost, Agarwal et al. [6] defined another iteration called S-iteration which is independent of Mann and Ishikawa iterations and it is defined by the sequence

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, T x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)\, x_n + \beta_n T x_n, \ n \geq 0, \end{aligned} \tag{8}$$

where $\alpha_n$ and $\beta_n$ are sequences in $[0, 1]$.

The S-iteration with errors can be given as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, T x_n + \alpha_n T y_n + e_n, \\ y_n &= (1-\beta_n)\, x_n + \beta_n T x_n + f_n, \ n \geq 0, \end{aligned} \tag{9}$$

where $e_n$ and $f_n$ satisfy $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\sum_{n=1}^{\infty} \|f_n\| < \infty$.

In [7], Thianwan defined a new iteration process given by the sequence

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, y_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)\, x_n + \beta_n T x_n, \ n \geq 0, \end{aligned} \tag{10}$$

where $\alpha_n$ and $\beta_n$ are sequences in $[0, 1]$.

The Thianwan iteration with errors is given by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, y_n + \alpha_n T y_n + e_n, \\ y_n &= (1-\beta_n)\, x_n + \beta_n T x_n + f_n, \ n \geq 0, \end{aligned} \tag{11}$$

where $e_n$ and $f_n$ satisfy $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\sum_{n=1}^{\infty} \|f_n\| < \infty$.

Khan [8] gave a different perspective to iteration procedure. He introduced the following hybrid Picard-Mann iterative scheme for a single nonexpansive mapping $T$ which is defined as

$$x_{n+1} = T y_n, \quad y_n = (1-\alpha_n)\, x_n + \alpha_n T x_n, \ n \geq 0, \tag{12}$$

where $\alpha_n$ is a sequence in $[0, 1]$ with $\sum \alpha_n = \infty$. While the Picard-Mann iteration process with errors is given as

$$\begin{aligned} x_{n+1} &= T y_n + e_n, \\ y_n &= (1-\alpha_n)\, x_n + \alpha_n T x_n + f_n, \ n \geq 0, \end{aligned} \tag{13}$$

where $e_n$ and $f_n$ satisfy $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\sum_{n=1}^{\infty} \|f_n\| < \infty$.

The most generalized operator used to approximate fixed point is the one proved by Zamfirescu [9]. The Zamfirescu operator was obtained from the Banach [2], Kannan [10], and Chatterjea [11] contractive mappings as follows. The operator $T$ is called a Kannan mapping if there exists $b \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)] \quad \forall x, y \in X. \tag{14}$$

Another similar definition due to Chatterjea mapping is as follows: there exists $c \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)] \quad \forall x, y \in X. \tag{15}$$

By combining (3), (14), and (15) conditions, we have the Zamfirescu operator given, for $x, y \in X$, by

$$\begin{aligned} &z_1\colon d(Tx, Ty) \leq a\, d(x, y), \quad a \in (0, 1), \\ &z_2\colon d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)], \\ &\qquad\qquad b \in \left(0, \frac{1}{2}\right), \\ &z_3\colon d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)], \\ &\qquad\qquad c \in \left(0, \frac{1}{2}\right). \end{aligned} \tag{16}$$

The equivalence of (16) is given as follows:

$$\begin{aligned} d(Tx, Ty) \leq h \max\{&d(x, y), d(x, Tx), d(y, Ty), \\ &d(x, Ty), d(y, Tx)\}, \end{aligned} \tag{17}$$

where $h = \max\{a, b, c\}$ and $x, y \in X$.

The following Lemma is useful in the proof of our results.

**Lemma 2** (see [12]). *Let $\delta$ be a real number such that $\delta \in [0, 1)$ and $\{u_n\}_{n=0}^{\infty}$ is a sequence of nonnegative numbers such that $\lim_{n\to\infty} u_n = 0$; then for any sequence of positive numbers $\{x_n\}_{n=0}^{\infty}$ satisfying*

$$x_{n+1} \leq \delta x_n + u_n \quad \forall n \in N, \tag{18}$$

*we have*

$$\lim_{n\to\infty} x_n = 0. \tag{19}$$

The Zamfirescu operator was used to prove the strong convergence of (1) as follows.

**Theorem 3.** *Let $C$ be a nonempty subset of a normed space $(E, \|\cdot\|)$. Let $T : C \to C$ be z-operator. If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\|u_n\| = 0$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* By Lemma 2, $T$ has a unique fixed point in $C$, say $q$. Let $x, y \in C$; since $T$ is a $z$-operator, at least one of each conditions $(z_1$–$z_3)$ is satisfied. If $z_2$ holds, then

$$d(Tx, Ty) \leq b\left[d(x, Tx) + d(y, Ty)\right] \tag{20}$$

is equivalent to

$$\begin{aligned}\|Tx - Ty\| &\leq b\left(\|x - Tx\| + \|y - Ty\|\right)\\ &= b\left(\|x - Tx\| + \|y - x + x - Tx + Tx - Ty\|\right)\\ &\leq b\left[\|x - Tx\| + \|y - x\| + \|x - Tx\| + \|Tx - Ty\|\right].\end{aligned} \tag{21}$$

This implies

$$(1 - b)\|Tx - Ty\| \leq b\|x - y\| + 2b\|x - Tx\|. \tag{22}$$

Therefore,

$$\|Tx - Ty\| \leq \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - Tx\|. \tag{23}$$

Similarly, if $z_3$ holds, we obtain

$$\|Tx - Ty\| \leq \frac{c}{1-c}\|x - y\| + \frac{2c}{1-c}\|x - Tx\|. \tag{24}$$

Let

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{25}$$

Then we have $0 \leq \delta < 1$ and in view of (25) it results in that inequalities (23) and (24) become

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\| \quad \forall x, y \in C. \tag{26}$$

Now, for $x = q$ and $y = x_n$ in (26), we obtain

$$\|Tq - Tx_n\| \leq \delta\|q - x_n\|, \tag{27}$$

which implies

$$\|x_{n+1} - q\| \leq \delta\|x_n - q\|. \tag{28}$$

By Lemma 2, we conclude that $x_n$ converges strongly to $q$. □

## 3. Analytical Results

We present our main results using $z$-operators (16) on normed spaces for the two-step iterative schemes with errors defined in the last section.

Throughout, $N$ denotes the set of all positive integers, $C$ is nonempty convex subset of a complete normed space $(E, \|\cdot\|)$, and $T : C \to C$ is a self-map.

**Theorem 4.** *Let $C$ be a nonempty subset of a normed space $(E, \|\cdot\|)$. Let $T : C \to C$ be z-operator (16). Let $\{x_n\}$ be defined with iterative process (5). If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\|u_n\| = 0$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* Let $x_0 \in C$; by using (5) and (16), we have

$$\begin{aligned}\|x_{n+1} - q\| &= \|(1 - \alpha_n)x_n + \alpha_n Tx_n + u_n - q\|\\ &= \|(1 - \alpha_n)x_n + \alpha_n Tx_n + u_n - q - \alpha_n q + \alpha_n q\|\\ &= \|(1 - \alpha_n)(x_n - q) + \alpha_n(Tx_n - q) + u_n\|\\ &\leq \|(1 - \alpha_n)(x_n - q)\| + \|\alpha_n(Tx_n - q)\| + \|u_n\|\\ &\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n\|Tx_n - q\| + \|u_n\|.\end{aligned} \tag{29}$$

Substituting (28) in (29),

$$\begin{aligned}\|x_{n+1} - q\| &\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n\delta\|x_n - q\| + \|u_n\|\\ &= (1 - \alpha_n + \alpha_n\delta)\|x_n - q\| + \|u_n\|\\ &= [1 - (1 - \delta)\alpha_n]\|x_n - q\| + \|u_n\|,\\ &\qquad n = 0, 1, 2, \ldots.\end{aligned} \tag{30}$$

Hence, by Lemma 2 and using the fact that $0 \leq \delta < 1$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\|u_n\| = 0$, it results in that

$$\lim_{n\to\infty}\|x_{n+1} - q\| = 0. \tag{31}$$

Therefore,

$$x_{n+1} \longrightarrow q \in F(T). \tag{32}$$

□

**Theorem 5.** *Let $C$ be a nonempty subset of a normed $(E, \|\cdot\|)$. Let $T : C \to C$ be z-operator (16). Let $\{x_n\}$ be defined with iterative process (13). If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\|g_n\| = 0$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* From (26), we have

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\| \quad \forall x, y \in C. \tag{33}$$

Also from (28), we have

$$\|Tx_n - q\| \leq \delta\|x_n - q\|. \tag{34}$$

Now, using (13) and (28), we have

$$\|x_{n+1} - q\| \leq \|Ty_n - q + e_n\| \leq \|Ty_n - q\| + \|e_n\|. \tag{35}$$

This implies

$$\|x_{n+1} - q\| \leq \delta\|y_n - q\| + \|e_n\|. \tag{36}$$

Also,

$$\begin{aligned}\|y_n - q\| &\leq \|(1 - \alpha_n)x_n + \alpha_n Tx_n - q + f_n\|\\ &\leq \|(1 - \alpha_n)x_n + \alpha_n Tx_n - q - \alpha_n q + \alpha_n q + f_n\|\\ &= \|(1 - \alpha_n)(x_n - q)\| + \alpha_n\|Tx_n - q\| + \|f_n\|\\ &\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n\|Tx_n - q\| + \|f_n\|.\end{aligned} \tag{37}$$

Substituting (37) in (36), we have

$$\begin{aligned}\|x_{n+1}-q\| &\le \delta\left[(1-\alpha_n)\|x_n-q\|+\alpha_n\|Tx_n-q\|\right]\\ &\quad+\|f_n\|+\|e_n\|.\end{aligned}\tag{38}$$

Using (28) in (38), we have

$$\begin{aligned}\|x_{n+1}-q\| &\le \delta\left[(1-\alpha_n)\|x_n-q\|+\alpha_n\delta\|x_n-q\|\right]\\ &\quad+\|f_n\|+\|e_n\|\\ &=\left[\delta-\alpha_n\delta\|x_n-q\|+\alpha_n\delta^2\|x_n-q\|\right]\\ &\quad+\|f_n\|+\|e_n\|\\ &=\left[\delta-\alpha_n\delta+\alpha_n\delta^2\right]\|x_n-q\|+\|f_n\|+\|e_n\|\\ &=\delta\left(1-(1-\delta)\alpha_n\right)\|x_n-q\|+\|g_n\|\\ &\le\left(1-(1-\delta)\alpha_n\right)\|x_n-q\|+\|g_n\|,\\ &\qquad n=0,1,2,\ldots,\end{aligned}\tag{39}$$

where $\|g_n\| \equiv \|f_n\| + \|e_n\|$.

By Lemma 2 and using the fact that $0 \le \delta < 1$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\|g_n\| = 0$, this becomes

$$\lim_{n\to\infty}\|x_{n+1}-q\|=0.\tag{40}$$

Hence,

$$x_{n+1}\longrightarrow q\in F(T).\tag{41}$$

□

**Theorem 6.** *Let $C$ be a nonempty subset of a normed space $(E,\|\cdot\|)$. Let $T: C\to C$ be z-operator. Let $\{x_n\}$ be defined with iterative process (7). If $F(T)\neq\phi$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\|g_n\|=0$. Then, $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* From (26), we have

$$\|Tx-Ty\|\le\delta\|x-y\|+2\delta\|x-Tx\|\quad\forall x,y\in C.\tag{42}$$

Also from (28), we have

$$\|Tx_n-q\|\le\delta\|x_n-q\|.\tag{43}$$

Now, by combining (7) and (28), we have

$$\begin{aligned}\|x_{n+1}-q\| &\le\|(1-\alpha_n)y_n+\alpha_nTx_n-q+e_n\|\\ &=\|(1-\alpha_n)(y_n-q)+\alpha_n(Tx_n-q)+e_n\|.\end{aligned}\tag{44}$$

This gives

$$\begin{aligned}\|x_{n+1}-q\| &\le(1-\alpha_n)\|y_n-q\|+\alpha_n\|Tx_n-q\|\\ &\quad+\|e_n\|.\end{aligned}\tag{45}$$

Also, from (7), we have

$$\begin{aligned}\|y_n-q\| &\le\|(1-\beta_n)x_n+\beta_nTx_n-q+f_n\|\\ &=\|(1-\beta_n)(x_n-q)+\beta_n(Tx_n-q)+f_n\|.\end{aligned}\tag{46}$$

This also gives

$$\|y_n-q\|\le(1-\beta_n)\|x_n-q\|+\beta_n\|Tx_n-q\|+\|f_n\|.\tag{47}$$

By combining (47) and (45), we have

$$\begin{aligned}&\|x_{n+1}-q\|\\ &\le(1-\alpha_n)\left[(1-\beta_n)\|x_n-q\|+\beta_n\|Tx_n-q\|\right]\\ &\quad+\alpha_n\|Tx_n-q\|+\|e_n\|+\|f_n\|\\ &=(1-\alpha_n)(1-\beta_n)\|x_n-q\|\\ &\quad+(1-\alpha_n)\beta_n\|Tx_n-q\|+\alpha_n\|Tx_n-q\|+\|e_n\|\\ &\quad+\|f_n\|\\ &\le(1-\alpha_n)(1-\beta_n)\|x_n-q\|\\ &\quad+\left[(1-\alpha_n)\beta_n+\alpha_n\right]\|Tx_n-q\|+\|e_n\|+\|f_n\|.\end{aligned}\tag{48}$$

Using (28) in (48), we have

$$\begin{aligned}&\|x_{n+1}-q\|\\ &\le(1-\alpha_n)(1-\beta_n)\|x_n-q\|\\ &\quad+\left[(1-\alpha_n)\beta_n+\alpha_n\right]\delta\|x_n-q\|+\|e_n\|+\|f_n\|\\ &=\left\{(1-\alpha_n)(1-\beta_n)+\left[(1-\alpha_n)\beta_n+\alpha_n\right]\delta\right\}\|x_n-q\|\\ &\quad+\|e_n\|+\|f_n\|\\ &=\left[(1-\alpha_n)(1-\beta_n)+(1-\alpha_n)\delta\beta_n+\alpha_n\delta\right]\|x_n-q\|\\ &\quad+\|e_n\|+\|f_n\|\\ &\le\left[1-(1-\delta)^2\alpha_n\right]\|x_n-q\|+\|e_n\|+\|f_n\|,\\ &\qquad n=0,1,2,\ldots.\end{aligned}\tag{49}$$

By letting $\|g_n\| = \|f_n\| + \|e_n\|$ and by Lemma 2, using the fact that $0 \le \delta < 1$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\|g_n\| = 0$, we get

$$\lim_{n\to\infty}\|x_{n+1}-q\|=0.\tag{50}$$

Therefore,

$$x_{n+1}\longrightarrow q\in F(T).\tag{51}$$

□

**Theorem 7.** *Let $C$ be a nonempty subset of a normed space $(E,\|\cdot\|)$. Let $T: C\to C$ be z-operator (16). Let $\{x_n\}$ be defined with iterative process (9). If $F(T)\neq\phi$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\|g_n\|=0$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* From (26), we have

$$\|Tx-Ty\|\le\delta\|x-y\|+2\delta\|x-Tx\|\quad\forall x,y\in C.\tag{52}$$

Also from (28), we have

$$\|Tx_n-q\|\le\delta\|x_n-q\|.\tag{53}$$

Table 1: The results for various iterations of Example 9.

| $n$ | Mann | Picard-Mann | Ishikawa | Thianwan | S-iteration |
|---|---|---|---|---|---|
| 0 | 1.990000000 | 1.990000000 | 1.990000000 | 1.990000000 | 1.990000000 |
| 1 | 1.687057835 | 1.572582235 | 1.813283737 | 1.809910017 | 1.582703394 |
| 2 | 1.580420632 | 1.527152378 | 1.703481407 | 1.699251872 | 1.52692284 |
| 3 | 1.542481649 | 1.522032849 | 1.63509312 | 1.631121832 | 1.52251127 |
| 4 | 1.52892988 | 1.521453636 | 1.592432357 | 1.58912183 | 1.521534023 |
| 5 | 1.524082164 | 1.521388075 | 1.565793786 | 1.563209142 | 1.521400741 |
| 6 | 1.522347139 | 1.521380654 | 1.549149294 | 1.547213668 | 1.521382574 |
| 7 | 1.521726048 | 1.521379814 | 1.53874517 | 1.537336782 | 1.521380098 |
| 8 | 1.525103699 | 1.521379719 | 1.532240114 | 1.531236794 | 1.52137976 |
| 9 | 1.521424097 | 1.521379708 | 1.528172254 | 1.527468964 | 1.521379714 |
| 10 | 1.521395599 | 1.521379707 | 1.52562821 | 1.525141481 | 1.521379708 |
| 11 | 1.521385396 | 1.521379707 | 1.524037063 | 1.523703668 | 1.521379707 |
| 12 | 1.521381741 | 1.521379707 | 1.523041857 | 1.522815428 | 1.521379707 |
| 13 | 1.521380435 | 1.521379707 | 1.522419375 | 1.522266655 | 1.521379707 |
| 14 | 1.521379968 | 1.521379707 | 1.522030019 | 1.521927661 | 1.521379707 |
| 15 | 1.52137980 | 1.521379707 | 1.521786478 | 1.521718232 | 1.521379707 |
| 16 | 1.521379740 | 1.521379707 | 1.521634144 | 1.521588448 | 1.521379707 |
| 17 | 1.521379731 | 1.521379707 | 1.521538858 | 1.521508914 | 1.521379707 |
| 18 | 1.521379715 | 1.521379707 | 1.521479257 | 1.521459531 | 1.521379707 |
| 19 | 1.521379710 | 1.521379707 | 1.521441976 | 1.521429016 | 1.521379707 |
| 20 | 1.521379708 | 1.521379707 | 1.521418656 | 1.52141017 | 1.521379707 |
| 21 | 1.521379707 | 1.521379707 | 1.52140407 | 1.521398527 | 1.521379707 |
| 22 | 1.521379707 | 1.521379707 | 1.521394946 | 1.521391334 | 1.521379707 |
| 23 | 1.521379707 | 1.521379707 | 1.521389239 | 1.52138689 | 1.521379707 |
| 24 | 1.521379707 | 1.521379707 | 1.521385669 | 1.521384145 | 1.521379707 |
| 25 | 1.521379707 | 1.521379707 | 1.521383436 | 1.521382449 | 1.521379707 |
| 26 | 1.521379707 | 1.521379707 | 1.52138204 | 1.521381401 | 1.521379707 |
| 27 | 1.521379707 | 1.521379707 | 1.521381166 | 1.521380753 | 1.521379707 |
| 28 | 1.521379707 | 1.521379707 | 1.521380620 | 1.521380353 | 1.521379707 |
| 29 | 1.521379707 | 1.521379707 | 1.521380278 | 1.521380106 | 1.521379707 |
| 30 | 1.521379707 | 1.521379707 | 1.521380064 | 1.521379954 | 1.521379707 |
| 31 | 1.521379707 | 1.521379707 | 1.521379930 | 1.521379860 | 1.521379707 |

Applying (28) to (9), we obtain

$$\begin{aligned}\|x_{n+1}-q\| &\le \|(1-\alpha_n)Tx_n+\alpha_nTy_n-q+e_n\|\\ &= \|(1-\alpha_n)(Tx_n-q)+\alpha_n(Ty_n-q)+e_n\|\\ &\le (1-\alpha_n)\delta\|x_n-q\|+\alpha_n\delta\|y_n-q\|+\|e_n\|\\ &\le (1-\alpha_n)\delta\|x_n-q\|+\alpha_n\delta\|y_n-q\|+\|e_n\|.\end{aligned} \tag{54}$$

Also, from (9), we have

$$\begin{aligned}\|y_n-q\| &\le \|(1-\beta_n)x_n+\beta_nTx_n-q+f_n\|\\ &= \|(1-\beta_n)(x_n-q)+\beta_n(Tx_n-q)+f_n\|\\ &\le (1-\beta_n)\|x_n-q\|+\beta_n\delta\|x_n-q\|+\|f_n\|\\ &\le [(1-\beta_n)+\beta_n\delta]\|x_n-q\|+\|f_n\|.\end{aligned} \tag{55}$$

Substituting (55) in (54), we have

$$\begin{aligned}\|x_{n+1}\| &\le [(1-\alpha_n)\delta+\alpha_n\delta((1-\beta_n)+\beta_n\delta)]\|x_n-q\|\\ &\quad +\|e_n\|+\|f_n\|.\end{aligned} \tag{56}$$

Inequality (56) becomes

$$\|x_{n+1}-q\| \le [1-(1-\delta)\alpha_n]\|x_n-q\|+\|e_n\|+\|f_n\|, \quad n=0,1,2,\ldots. \tag{57}$$

By letting $\|g_n\| = \|f_n\| + \|e_n\|$ and by Lemma 2, using the condition of the theorem, we have

$$\lim_{n\to\infty}\|x_{n+1}-q\| = 0. \tag{58}$$

Hence,

$$x_{n+1} \longrightarrow q \in F(T). \tag{59}$$

□

Table 2: The results for various iterations of Example 10.

| $n$ | Mann | Picard-Mann | Ishikawa | Thianwan | S-iteration |
|---|---|---|---|---|---|
| 0 | 4.000000000 | 4.000000000 | 4.000000000 | 4.000000000 | 4.000000000 |
| 1 | 3.658312395 | 3.211950309 | 3.316624790 | 3.210186229 | 3.210186229 |
| 2 | 3.360737671 | 3.05436713 | 3.174706708 | 3.10855584 | 3.061581272 |
| 3 | 3.178630457 | 3.015059648 | 3.11619388 | 3.070267725 | 3.019321892 |
| 4 | 3.082897312 | 3.004347136 | 3.085110108 | 3.050716647 | 3.006219262 |
| 5 | 3.036738167 | 3.001287744 | 3.066152455 | 3.03905625 | 3.002026325 |
| 6 | 3.015723592 | 3.000388341 | 3.053529001 | 3.031402901 | 3.000664646 |
| 7 | 3.006547498 | 3.000118657 | 3.044594314 | 3.026040224 | 3.000218903 |
| 8 | 3.002666794 | 3.000036622 | 3.037979605 | 3.022099423 | 3.000072291 |
| 9 | 3.001066599 | 3.000011394 | 3.032910338 | 3.019096558 | 3.000023918 |
| 10 | 3.000420156 | 3.000003568 | 3.028917572 | 3.016742078 | 3.000007924 |
| 11 | 3.000163391 | 3.000001123 | 3.025701936 | 3.014852836 | 3.000002628 |
| 12 | 3.000062842 | 3.000000355 | 3.023063963 | 3.013307708 | 3.000000872 |
| 13 | 3.00002394 | 3.000000113 | 3.020865992 | 3.012023608 | 3.000000029 |
| 14 | 3.000009044 | 3.000000036 | 3.019010164 | 3.010941775 | 3.000000001 |
| 15 | 3.00000331 | 3.000000012 | 3.017425149 | 3.010019562 | 3.000000000 |
| 16 | 3.000001263 | 3.000000004 | 3.016057811 | 3.009225316 | 3.000000000 |
| 17 | 3.000000468 | 3.000000001 | 3.014867814 | 3.00853509 | 3.000000000 |
| 18 | 3.000000172 | 3.000000000 | 3.013824028 | 3.007930452 | 3.000000000 |
| 19 | 3.000000063 | 3.000000000 | 3.012902073 | 3.007397005 | 3.000000000 |
| 20 | 3.000000023 | 3.000000000 | 3.012082600 | 3.006923067 | 3.000000000 |
| 21 | 3.000000008 | 3.000000000 | 3.011350076 | 3.006500078 | 3.000000000 |
| 22 | 3.000000003 | 3.000000000 | 3.010691894 | 3.006120341 | 3.000000000 |
| 23 | 3.000000001 | 3.000000000 | 3.010097723 | 3.005777802 | 3.000000000 |
| 24 | 3.000000000 | 3.000000000 | 3.009559027 | 3.005467464 | 3.000000000 |
| 25 | 3.000000000 | 3.000000000 | 3.009068691 | 3.00518517 | 3.000000000 |
| 26 | 3.000000000 | 3.000000000 | 3.008620741 | 3.004927435 | 3.000000000 |
| 27 | 3.000000000 | 3.000000000 | 3.008210132 | 3.004691315 | 3.000000000 |
| 28 | 3.000000000 | 3.000000000 | 3.007832568 | 3.004474311 | 3.000000000 |
| 29 | 3.000000000 | 3.000000000 | 3.007484378 | 3.004274286 | 3.000000000 |
| 30 | 3.000000000 | 3.000000000 | 3.007162403 | 3.004089403 | 3.000000000 |
| 31 | 3.000000000 | 3.000000000 | 3.00686391 | 3.003918076 | 3.000000000 |

**Theorem 8.** *Let $C$ be a nonempty subset of a normed space $(E, \|\cdot\|)$. Let $T : C \to C$ be z-operator (16). Let $\{x_n\}$ be defined with iterative process (11). If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\|u_n\| = 0$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* From (26), we have

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| \quad \forall x, y \in C. \tag{60}$$

Also from (28), we have

$$\|Tx_n - q\| \leq \delta \|x_n - q\|. \tag{61}$$

Now, applying (28) to $x_{n+1}$ in (11), we get

$$\begin{aligned} \|x_{n+1} - q\| &\leq \|(1 - \alpha_n) y_n + \alpha_n T y_n - q + e_n\| \\ &= \|(1 - \alpha_n)(y_n - q) + \alpha_n (T y_n - q) + e_n\| \\ &\leq (1 - \alpha_n) \|y_n - q\| + \alpha_n \delta \|y_n - q\| + \|e_n\|. \end{aligned} \tag{62}$$

This implies

$$\|x_{n+1} - q\| \leq (1 - \alpha_n + \alpha_n \delta) \|y_n - q\| + \|e_n\|. \tag{63}$$

Also, applying (28) to $y_n$ in (11), we get

$$\begin{aligned} \|y_n - q\| &\leq \|(1 - \beta_n) x_n + \beta_n T x_n - q + f_n\| \\ &= \|(1 - \beta_n)(x_n - q) + \beta_n (T x_n - q) + f_n\| \\ &\leq (1 - \beta_n) \|x_n - q\| + \beta_n \delta \|x_n - q\| + \|f_n\|. \end{aligned} \tag{64}$$

This gives

$$\|y_n - q\| \leq (1 - \beta_n + \beta_n \delta) \|x_n - q\| + \|f_n\|. \tag{65}$$

By combining (63) and (65), we have

$$\begin{aligned} &\|x_{n+1} - q\| \\ &\leq (1 - \alpha_n + \alpha_n \delta)(1 - \beta_n + \beta_n \delta) \|x_n - q\| + \|e_n\| \\ &\quad + \|f_n\| \end{aligned}$$

Table 3: The error estimation for various iterations of Example 9.

| $n$ | Error estimation ($e_n$) | | | | |
|---|---|---|---|---|---|
| | Mann | Picard-Mann | Ishikawa | Thianwan | S-iteration |
| 5 | 0.002702457 | 0.000008368 | 0.044414079 | 0.041829435 | 0.000021034 |
| 10 | 0.000015892 | 0.0 | 0.004248503 | 0.003761774 | 0.000000001 |
| 15 | 0.00000093 | 0.0 | 0.000406771 | 0.000338525 | 0.0 |
| 20 | 0.000000001 | 0.0 | 0.000038949 | 0.000030463 | 0.0 |
| 25 | 0.0 | 0.0 | 0.000003729 | 0.000002742 | 0.0 |
| 30 | 0.0 | 0.0 | 0.000000357 | 0.000000247 | 0.0 |

Table 4: The error estimation for various iterations of Example 10.

| $n$ | Error estimation ($e_n$) | | | | |
|---|---|---|---|---|---|
| | Mann | Picard-Mann | Ishikawa | Thianwan | S-iteration |
| 5 | 0.036738167 | 0.001287744 | 0.066152455 | 0.03905625 | 0.002026325 |
| 10 | 0.000420156 | 0.000003568 | 0.028917572 | 0.016742078 | 0.000007924 |
| 15 | 0.000003391 | 0.000000012 | 0.017425149 | 0.010019562 | 0.0 |
| 20 | 0.000000023 | 0.0 | 0.012902073 | 0.006923067 | 0.0 |
| 25 | 0.0 | 0.0 | 0.009559027 | 0.00518517 | 0.0 |
| 30 | 0.0 | 0.0 | 0.007162403 | 0.004089403 | 0.0 |

$$= \left[1-(1-\delta)\alpha_n\right]\left[1-(1-\delta)\beta_n\right]\|x_n-q\| + \|e_n\| + \|f_n\| \le \left[1-(1-\delta)\alpha_n\right]\|x_n-q\| + \|g_n\|, \quad n=0,1,2,\ldots. \tag{66}$$

Thus, applying Lemma 2 and the condition of the theorem, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-q\| = 0. \tag{67}$$

Therefore,

$$x_{n+1} \longrightarrow q \in F(T). \tag{68}$$

□

## 4. Numerical Results

In this section, we support our analytical results with the following two numerical examples.

*Example 9.* Let the function $f : [0,4] \to [0,4]$ be defined by $f(x) = (x+2)^{1/3}$ with fixed point $q = 1.521379707$. Initial guess is $x_0 = 1.99$, $\alpha_n = \beta_n = 1/4$.

*Example 10.* Let the function $f : [0,4] \to [0,4]$ be defined by $f(x) = (2x+3)^{1/2}$ with fixed point $q = 3.000000000$. Initial guess is $x_0 = 4$, $\alpha_n = 1/(n+2)$, $\beta_n = 1/(n+1)$.

The results for Examples 9 and 10 with various iterations are presented in Tables 1 and 2, respectively.

The error estimation ($e_n$) in Table 1 choosing $n$ at an interval of 5 iterations is presented in Table 3, where $e_n = \|q - x_n\|$.

The calculated error in Table 2 choosing $n$ at an interval of 5 iterations is presented in Table 4.

## 5. Conclusion

This paper proved the convergence and convergence rate of some two-step iterative schemes with errors using Zamfirescu operator in Banach spaces. It was observed from Example 9 that the Picard-Mann iteration converges faster than the S-iteration, Ishikawa iteration, and Thianwan iteration, while, in Example 10, it is seen that the S-iteration has better convergence rate than other iterations. This is so because the speed of the aforementioned iterations depends on the choices of the parametric constants $\alpha_n$ and $\beta_n$.

## Disclosure

Authors agreed to be accountable for all aspects of the work in ensuring that questions related to the accuracy or integrity of any part of the work are appropriately investigated and resolved.

## Competing Interests

The authors hereby declare that there are no competing interests.

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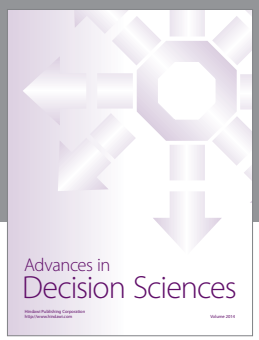
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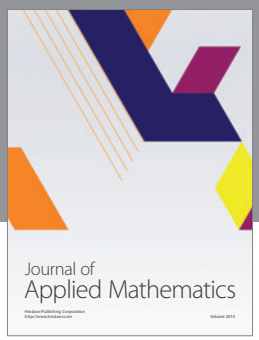
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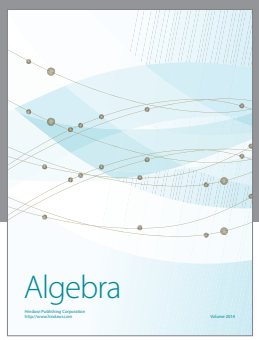
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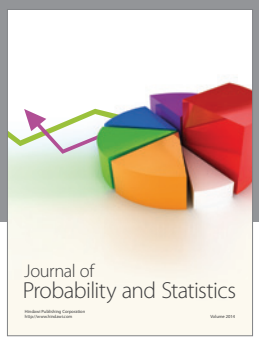
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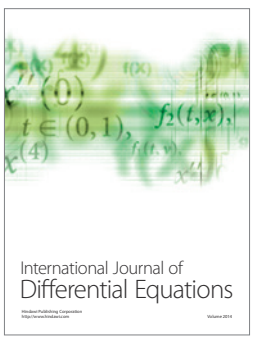


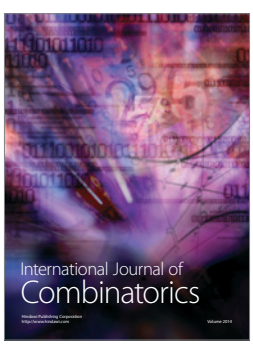


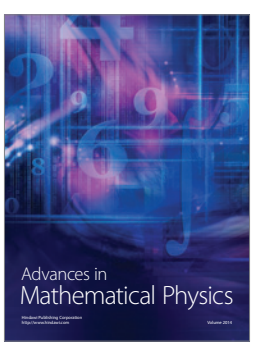


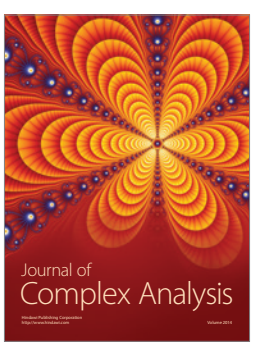


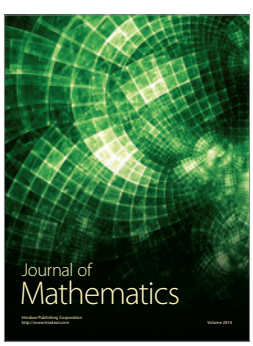


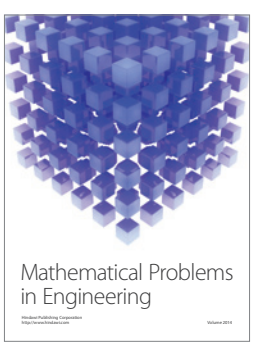


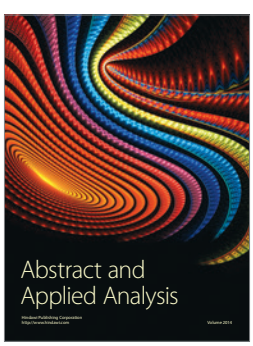


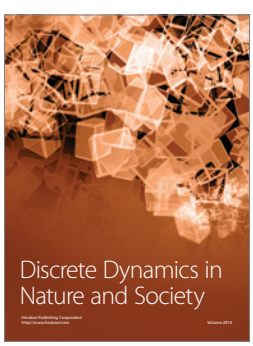


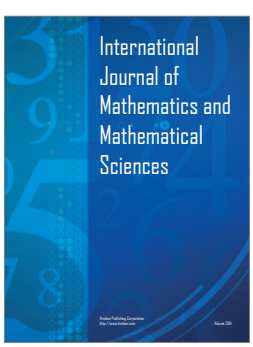






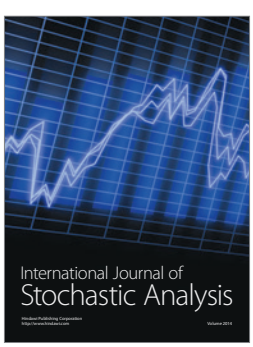














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