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SOME RESULTS ON COMMON FIXED POINT FOR GENERALIZED f-CONTRACTION MAPPING

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Abstract

In this article, we prove some common fixed point theorems for the generalized f-contraction mapping in a complete cone metric space. Our results extend and generalized some recent results.

Keywords: Common Fixed Point, f-Contraction Mapping, complete cone metric space.

Subject Classification: 47H10, 54H25.

1. Introduction

The first important result on fixed point for contractive type mapping was the well known Banach's contraction principle appeared in 1922 where it was used to establish the existence of a solution for an integral equation [1]. It is well known that the classic contraction mapping of Banach is a fundamental result in fixed point theory, Several authors have obtained various extension, refinement and generalization of Banach's theorem by considering contractive mapping on many different metric spaces.

Edelstein [8] considered a non-expansive contractive type mappings. [2] defined a contractive mapping with the sum of three non-negative numbers less than one. [9] proved that every continuous self-map on the metric space (X,d) which satisfies the condition

$$d(Tx,Ty) \le \frac{\alpha d(y,Ty)(1+d(x,Tx))}{1+d(x,y)} + \beta d(x,y)$$

for all $x, y \in X$ where $\alpha, \beta \in [0,1)$ and $\alpha + \beta < 1$ have a unique fixed point. [19] compared 250 types of contractive definitions and analyzed the relationship among them.

In [12], Huang and Zhuang introduced the notion of cone metric space, they have generalized the concept of metric space, replacing the set real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions on a complete cone metric space with the assumption of normality of a cone. These authors also described the convergence of sequences in the cone metric spaces and introduced the corresponding notion of completeness.

[3] introduced new classes of contractive functions called f-contraction and f-contractive mapping and then established and extended the Banach contraction mapping and Edelstein's fixed point theorems. [16] obtained sufficient condition for the existence of a unique fixed point of f-Kanna contractive, f-Zanfirescu, f-contractive mapping. On complete cone metric spaces. Sumitra et al. [20] proved common fixed point theorem for a Banach pair of mappings satisfying f-Hardy Rogers type contraction condition in cone metric spaces. [6] extended fixed point theory of weak contraction to cone metric spaces. [17] proved some common fixed point theorems for f-contraction mappings in cone metric spaces. Chandok and Narang in [4] and [5] introduced common fixed points and invariant approximation for Banach operators pairs with Ciric type of non-expansive mappings [7]. In [18] Rangamma and Prudhvi proved the existence of coincidence points and a common fixed theorem for four maps in cone metric spaces. [15] derived some common fixed point results for Banach operator pairs in strongly M-starshaped metric spaces. [10] and [11] also used the T-Zamfirescu mappings to obtain the generalized cone metric spaces and fixed point theorems. For further recent work, see [13] and [14]. Several useful extensions and generalizations could be sourced from [21].

In this paper, we establish a new modified f-Hardy Roger contractive mapping of two Banach pairs of mappings to obtain a common fixed point results.

2. Preliminaries

In this section, we give some definitions and notations on cone metric spaces and their properties.

Definition 2.1 *Öztürk, M. and Metin, B. (2011)*

Let E be a real Banach space and K be a subset of E. K is called a cone if and only if

- K is closed, nonempty and $K \neq \{0\}$
- $ax + by \in K$ for $x, y \in K$ and $a, b \geq 0$
- $x \in K$ and $-x \in K \Rightarrow x = 0 \Leftrightarrow K \cap (-K) = \{0\}$

Consider a cone $K \subset E$. We define a partial ordering "$\leq$" with respect to K by $x \leq y$ if and only if $y - x \in K$, we write $x < y$ to indicate that $x \leq y$ but $x \neq y$ and $x << y$ to imply $y - x \in intK$. The $intK$ denotes the interior of K.

A cone K is called a normal if for a given number $M > 0$ such that for all $x, y \in K$

$$0 \leq x \leq y \Rightarrow \|x\| \leq M\|y\| \tag{1}$$

The least positive number of M satisfying (1) is called the normal constant of K.

The cone K is called regular if for every $n \in \mathbb{N}$ such that $x_n \leq x_{n+1} \leq y$ for some $y \in E$, then there is $x \in E$ such that

$$\lim_{n \to \infty} \|x_n - x\| = 0$$

i.e. K is called regular if every monotone increasing sequence bounded from above is convergent.

Similarly, K is called regular if every monotone decreasing sequence bounded from below is convergent.

Note: Every regular cone is a normal cone.

Definition 2.2 *Öztürk, M. and Metin, B. (2011)*

Let X be a nonempty set and $E \supset K$ be a real Banach spaces. Suppose the metric mapping $d : X \times X \to E$ satisfies

- $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X and (X, d) is called a cone metric space

Example 2.3 *Let* $E = R^3$, *define* $K = \{(x, y, z) \in E : x, y, z \geq 0\} \subset R^3$.

Let $X = R$ and define $d : X \times X \to E$ such that

$$d(x_1, x_2) = \left(\alpha|x_1 - x_2|, \beta|x_1 - x_2|, \gamma|x_1 - x_2|\right)$$

where $x_1, x_2 \in X$ and α, β, γ are nonnegative.

Then, d is a cone metric on X and (X, d) is a cone metric space.

Lemma 2.4 *Dubey et al (2013)*

Let (X,d) be a cone metric space and $\{x_n\}$ be a sequence in X, if for every $c \in E$ with $0 << c$

- $\{x_n\}$ converges to x, then given $n(c) \in \mathbb{N}$ such that $\|d(x_n,x)\| << c$ for all $n > n(c)$.
- $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $\|d(x,y)\| << c \Rightarrow x = y$.
- $\{x_n\}$ is Cauchy sequence, then there is $n(c) \in \mathbb{N}$ such that $\|d(x_n,x_m)\| << c$ for $n,m > n(c)$.
- $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively, then for given $n(c) \in \mathbb{N}$ and $n > n(c)$, $\|d(x_n,y_n) - d(x,y)\| \to 0$.
- $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy sequence.
- the Cauchy sequence converges to a point in X, then (X,d) is complete.

Lemma 2.5 *Dubey et al (2014)*

Let (X,d) be a cone metric space and $u,v,w \in X$.

- if $u << v$ and $v << w$, then $u << w$
- if $u \le v$ and $v << w$, then $u << w$
- if $0 \le u << c$ for each $c \in intK$, then $u = 0$
- if $0 \le u << \alpha c$ for $\alpha > o$ and each $c \in intK$, then $u = 0$
- if $c \in intK$, $0 \le u_n$ and $u_n \to 0$, then there is $n(c)$ such that $u_n << c$ for all $n > n(c)$.

Definition 2.6 *Banach (1922)*

A self mapping T of a metric space (X,d) is said to be a contraction mapping if there exists a number $k \in [0,1)$ such that

$$d(Tx,Ty) \le kd(x,y), \quad for\ all \quad x,y \in X. \tag{2}$$

Definition 2.7 *Beiranvand et al (2009)*

Let (X,d) be a metric space and $f,T : X \to X$ be two self mappings. The mapping T is called f -contraction if there a number $k \in [0,1)$ such that

$$d(fTx, fTy) \le kd(fx, fy), \quad for\ all \quad x,y \in X. \tag{3}$$

If f is an identity mapping, then (3) reduces to (2).

Definition 2.8 *Let (X,d) be a metric space and $f,T,S : X \to X$ be three self mappings. The pair (T,S) is called f -contraction if there a number $k \in [0,1)$ such that*

$$d(fTx, fSy) \le kd(fx, fy), \quad for\ all \quad x,y \in X. \tag{4}$$

The following example is given to justify our claim in Definition 2.8

Example 2.9 *Let $[0,\infty)$ be endowed with the usual metric and let the mappings be defined by*

$$fx = e^{-x},\ Tx = 2x + \ln 2^{\beta},\ Sy = 2y + \ln 2^{(\beta-1)}$$

where x, y are partially ordered in X and $\beta > 1$

Clearly, the pair (T, S) is not a contraction. But

$$\begin{aligned} d(fTx, Sfy) &= \left|2^{-\beta} e^{-2x} - 2^{-(\beta-1)} e^{-2(y-1)}\right| \\ &\le \left|2^{-(\beta-1)}\right| e^{-2x} - e^{-2(y-1)}\Big| \\ &\le 2^{-(\beta-1)} \left|e^{-2x} - e^{-2y}\right| \\ &\le 2^{-(\beta-1)} \left|e^{-x} - e^{-y}\right| \\ &\equiv k\left|fx - fy\right|, \; k \equiv 2^{-(\beta-1)} < 1 \end{aligned}$$

Therefore, the pair (T, S) is f -contraction. We end this section with the following crucial lemma

Lemma 2.10 *Öztürk, M. and Metin, B. (2011)*

Let (X, d) be a cone metric space, then

• a mapping $T : X \to X$ is continuous if for $n \in \mathbb{N}$ $\lim_{n\to\infty} x_n \to x$ implies that $\lim_{n\to\infty} Tx_n \to Tx$ for $\{x_n\}$ and x are in X .

• the mapping $f, T : X \to X$ or $f, S : X \to X$ is said to be sequentially convergent, if the sequence $\{y_n\}$ in X is convergent whenever $\{Ty_n\}$ is convergent.

• the mapping $f, T : X \to X$ or $f, S : X \to X$ is said to be subsequentially convergent, if the sequence $\{y_n\}$ in X has a convergent subsequence whenever $\{Ty_n\}$ is convergent.

3. Main Results

In this section, we give our results which are generalization of some existing results.

Theorem 3.1 *Let (X, d) be a nonempty cone metric space and let f, T, S be three continuous self mapping on a complete cone metric space (X, d). Assume f is an injective mapping and K is a normal cone with normal constant M. Suppose f, T, S satisfy*

$$d(fTx, fSy) \le \alpha d(fx, fy) + \beta\left[d(fTx, fx) + d(fSy, fy)\right] + \gamma\left[d(fTx, fy) + d(fSy, fx)\right] \qquad (5)$$

where $\alpha + 2\beta + 2\gamma < 1$, for $\alpha, \beta, \gamma \in [0,1)$ and $x, y \in X$. Then S and T have a unique common fixed point in X . Moreover, if (f, T) and (f, S) are Banach pairs, then f, T and S have a unique common fixed point in X.

Proof

Let x_o be fixed and define $x_{2n} = Tx_{2n-1}$ and $x_{2n+1} = Sx_{2n}$.

Consider

$$\begin{aligned} d(fx_{2n}, fx_{2n+1}) &= d(fTx_{2n-1}, fSx_{2n}) \\ &\le \alpha d(fx_{2n-1}, fx_{2n}) + \beta\left[d(fTx_{2n-1}, fx_{2n-1}) + d(fSx_{2n}, fx_{2n})\right] \\ &\quad + \gamma\left[d(fTx_{2n-1}, fx_{2n}) + d(fSx_{2n}, fx_{2n-1})\right] \\ &= \alpha d(fx_{2n-1}, fx_{2n}) + \beta d(fx_{2n}, fx_{2n-1}) + \beta d(fx_{2n+1}, fx_{2n}) \\ &\quad + \gamma d(fx_{2n}, fx_{2n}) + \gamma d(fx_{2n+1}, fx_{2n-1}) \\ &\le \alpha d(fx_{2n-1}, fx_{2n}) + \beta d(fx_{2n}, fx_{2n-1}) + \beta d(fx_{2n+1}, fx_{2n}) \\ &\quad + \gamma d(fx_{2n}, fx_{2n-1}) \end{aligned}$$

$$\Rightarrow (1-\beta)d(fx_{2n}, fx_{2n+1}) \le (\alpha+\beta+\gamma)d(fx_{2n-1}, fx_{2n}) \tag{6}$$

Similarly,

$$\begin{aligned} d(fx_{2n+1}, fx_{2n}) &= d(fSx_{2n}, fTx_{2n-1}) \\ &\le \alpha d(fx_{2n}, fx_{2n-1}) + \beta\left[d(fSx_{2n}, fx_{2n}) + d(fTx_{2n-1}, fx_{2n-1})\right] \\ &\quad + \gamma\left[d(fSx_{2n}, fx_{2n-1}) + d(fTx_{2n-1}, fx_{2n})\right] \\ &= \alpha d(fx_{2n}, fx_{2n-1}) + \beta d(fx_{2n+1}, fx_{2n}) + \beta d(fx_{2n}, fx_{2n-1}) \\ &\quad + \gamma d(fx_{2n+1}, fx_{2n-1}) + \gamma d(fx_{2n}, fx_{2n}) \\ &\le \alpha d(fx_{2n}, fx_{2n-1}) + \beta d(fx_{2n+1}, fx_{2n}) + \beta d(fx_{2n}, fx_{2n-1}) \\ &\quad + \gamma d(fx_{2n+1}, fx_{2n}) \end{aligned}$$

$$\Rightarrow (1-\beta-\gamma)d(fx_{2n+1}, fx_{2n}) \le (\alpha+\beta)d(fx_{2n}, fx_{2n-1}) \tag{7}$$

By adding (6) and (7), we have

$$d(fx_{2n}, fx_{2n+1}) \le \frac{2\alpha+2\beta+\gamma}{2-2\beta-\gamma} d(fx_{2n-1}, fx_{2n}) \tag{8}$$

If we let $k = \dfrac{2\alpha+2\beta+\gamma}{2-2\beta-\gamma} < 1$

$$\Rightarrow d(fx_{2n}, fx_{2n+1}) \le kd(fx_{2n-1}, fx_{2n})$$

Also,

$$d(fx_{2n-1}, fx_{2n}) \le kd(fx_{2n-2}, fx_{2n-1})$$

In general,

$$d(fx_{2n}, fx_{2n+1}) \le k^{2n} d(fx_0, fx_1)$$

Let $m > n$ for $m, n \in \mathbb{N}$, by triangular inequality, we have

$$\begin{aligned} d(fx_{2n}, fx_{2m}) &\le d(fx_{2n}, fx_{2n+1}) + d(fx_{2n+1}, fx_{2n+2}) + \ldots + d(fx_{2m-1}, fx_{2m}) \\ &\le k^{2n} d(fx_0, fx_1) + k^{2n+1} d(fx_0, fx_1) + \ldots + k^{2m-1} d(fx_0, fx_1) \\ &= k^{2n}(1 + k + k^2 + \ldots + k^{2m-2n-1}) d(fx_0, fx_1) \\ &= k^{2n}\left(\frac{1-k^{2m-2n}}{1-k}\right) d(fx_0, fx_1) \\ &\le \frac{k^{2n}}{1-k} d(fx_0, fx_1), \quad \text{for } 1-k^{2m-2n} < 1 \end{aligned}$$

From (1), we obtain

$$\left\| d(fx_{2n}, fx_{2m}) \right\| \le \left| \frac{k^{2n}}{1-k} \right| M \left\| d(fx_0, fx_1) \right\| \tag{9}$$

Since $k < 1$ and $d(fx_0, fx_1)$ is fixed, then $k^{2n} \to 0$ as $n \to \infty$. And by lemma 2.5 (iv)-(v), we have

$$d(fx_{2n}, fx_{2m}) << c$$

Thus, fx_{2n} is a Cauchy sequence in X. Since X is a complete cone metric space, fx_{2n} converges to a point (say t

) in X . Since f is subsequentially convergent, then x_{2n} has a subsequence x_{2m} which also converges to a point (say u). Since f is continuous, then

$$\lim_{m\to\infty} fx_{2m} = fu \tag{10}$$

By the uniqueness of the limit $t = fu$.

Since T and S are continuous, then

$$\lim_{m\to\infty} fTx_{2m} = fTu \tag{11}$$

and

$$\lim_{m\to\infty} fSx_{2m} = fSu \tag{12}$$

Now, let m be odd, by triangular inequality and (5), we have

$$\begin{aligned} d(fu, fTu) &\le d(fu, fx_{2n+1}) + d(fx_{2n+1}, fTu) \\ &= d(fu, fx_{2n+1}) + d(fSx_{2n}, fTu) \\ &\le d(fu, fx_{2n+1}) + \alpha d(fx_{2n}, fu) + \beta\left[d(fSx_{2n}, fx_{2n}) + d(fTu, fu)\right] \\ &\quad + \gamma\left[d(fSx_{2n}, fu) + d(fTu, fx_{2n})\right] \\ &\le d(fu, fx_{2n+1}) + \alpha d(fx_{2n}, fu) + \beta d(fx_{2n+1}, fx_{2n}) + \beta d(fTu, fu) \\ &\quad + \gamma d(fx_{2n+1}, fu) + \gamma d(fTu, fu) + \gamma d(fu, fx_{2n}) \end{aligned}$$

This implies

$$d(fu, fTu) \le \frac{1+\gamma}{1-\beta-\gamma} d(fx_{2n+1}, fu) + \frac{\alpha+\gamma}{1-\beta-\gamma} d(fx_{2n}, fu) + \frac{\beta}{1-\beta-\gamma} d(fx_{2n}, fx_{2n+1}) \tag{13}$$

By applying the normality condition (1), we have

$$\begin{aligned} \|d(fu, fTu)\| &\le M\left\| \frac{1+\gamma}{1-\beta-\gamma} d(fx_{2n+1}, fu) + \frac{\alpha+\gamma}{1-\beta-\gamma} d(fx_{2n}, fu) \right. \\ &\quad \left. + \frac{\beta}{1-\beta-\gamma} d(fx_{2n}, fx_{2n+1}) \right\| \end{aligned}$$

$$\begin{aligned} \|d(fu, fTu)\| &\le M\left|\frac{1+\gamma}{1-\beta-\gamma}\right| \|d(fx_{2n+1}, fu)\| + M\left|\frac{\alpha+\gamma}{1-\beta-\gamma}\right| \|d(fx_{2n}, fu)\| \\ &\quad + M\left|\frac{\beta}{1-\beta-\gamma}\right| \|d(fx_{2n}, fx_{2n+1})\| \end{aligned} \tag{14}$$

we have from (10) that

$$\|d(fx_{2n+1}, fu)\| << \frac{c|1-\beta-\gamma|}{3M|1+\gamma|} \quad and \quad \|d(fx_{2n}, fu)\| << \frac{c|1-\beta-\gamma|}{3M|\alpha+\gamma|}$$

Therefore,

$$\begin{aligned}\|d(fu, fTu)\| &<< M\left|\frac{1+\gamma}{1-\beta-\gamma}\right|\frac{c|1-\beta-\gamma|}{3M|1+\gamma|}+M\left|\frac{\alpha+\gamma}{1-\beta-\gamma}\right|\frac{c|1-\beta-\gamma|}{3M|\alpha+\gamma|}\\ &\quad +M\left|\frac{\beta}{1-\beta-\gamma}\right|\left|\frac{c|1-\beta-\gamma|}{3M|\beta|}\right|\\ &=\frac{c}{3}+\frac{c}{3}+\frac{c}{3}=c\end{aligned}$$

Hence, by lemma 2.5,

$$\|d(fu, fTu)\|=0\Leftrightarrow fu=fTu$$

Also, let m be even

$$\begin{aligned}d(fu, fSu) &\le d(fu, fx_{2n})+d(fx_{2n}, fSu)\\ &= d(fu, fx_{2n})+d(fTx_{2n-1}, fSu)\\ &\le d(fu, fx_{2n})+\alpha d(fx_{2n-1}, fu)+\beta\left[d(fTx_{2n-1}, fx_{2n-1})+d(fSu, fu)\right]\\ &\quad +\gamma\left[d(fTx_{2n-1}, fu)+d(fSu, fx_{2n-1})\right]\\ &\le d(fu, fx_{2n})+\alpha d(fx_{2n-1}, fu)+\beta d(fx_{2n}, fx_{2n-1})+\beta d(fSu, fu)\\ &\quad +\gamma d(fx_{2n}, fu)+\gamma d(fSu, fu)+\gamma d(fu, fx_{2n-1})\end{aligned}$$

This implies

$$d(fu, fSu)\le\frac{1+\gamma}{1-\beta-\gamma}d(fx_{2n}, fu)+\frac{\alpha+\gamma}{1-\beta-\gamma}d(fx_{2n-1}, fu)+\frac{\beta}{1-\beta-\gamma}d(fx_{2n}, fx_{2n-1})$$

So,

$$\begin{aligned}\|d(fu, fSu)\| &\le M\left\|\frac{1+\gamma}{1-\beta-\gamma}d(fx_{2n}, fu)+\frac{\alpha+\gamma}{1-\beta-\gamma}d(fx_{2n-1}, fu)\right.\\ &\quad \left.+\frac{\beta}{1-\beta-\gamma}d(fx_{2n}, fx_{2n-1})\right\|\end{aligned}$$

$$\begin{aligned}\|d(fu, fSu)\| &\le M\left|\frac{1+\gamma}{1-\beta-\gamma}\right|\|d(fx_{2n}, fu)\|+M\left|\frac{\alpha+\gamma}{1-\beta-\gamma}\right|\|d(fx_{2n-1}, fu)\|\\ &\quad +M\left|\frac{\beta}{1-\beta-\gamma}\right|\|d(fx_{2n}, fx_{2n-1})\|\\ &<< M\left|\frac{1+\gamma}{1-\beta-\gamma}\right|\frac{c|1-\beta-\gamma|}{3M|1+\gamma|}+M\left|\frac{\alpha+\gamma}{1-\beta-\gamma}\right|\frac{c|1-\beta-\gamma|}{3M|\alpha+\gamma|}\\ &\quad +M\left|\frac{\beta}{1-\beta-\gamma}\right|\left|\frac{c|1-\beta-\gamma|}{3M|\beta|}\right|\\ &=\frac{c}{3}+\frac{c}{3}+\frac{c}{3}=c\end{aligned}$$

Also, by lemma 2.5,

$$\|d(fu, fSu)\|=0\Leftrightarrow fu=fSu$$

which implies that $fu=fTu=fSu$.

For uniqueness, suppose $fu = fTu = fSu$ and $f\bar{u} = fT\bar{u} = fS\bar{u}$ are two fixed points, then

$$\begin{aligned} d(fu, f\bar{u}) &= d(fTu, fS\bar{u}) \\ &\le \alpha d(fu, f\bar{u}) + \beta[d(fTu, fu) + d(fS\bar{u}, f\bar{u})] + \gamma[d(fTu, f\bar{u}) + d(fS\bar{u}, fu)] \\ &= \alpha d(fu, f\bar{u}) + \beta d(fu, fu) + \beta d(f\bar{u}, f\bar{u}) + \gamma d(fu, f\bar{u}) + \gamma d(f\bar{u}, fu) \\ &\le (\alpha + 2\gamma) d(fu, f\bar{u}) \\ &< (\alpha + 2\beta + 2\gamma) d(fu, f\bar{u}) \\ &< d(fu, f\bar{u}) \end{aligned}$$

This is a contradiction and hence

$d(fu, f\bar{u}) = 0$ if and only if $fu = f\bar{u}$

Since f is an injective, then $u = \bar{u}$ is the unique fixed point. Since we have assumed that (f, S) and (f, T) are Banach pairs, then (f, S) and (f, T) commute at the fixed point of S and T respectively. This implies that $fSu = Sfu$ for all $u \in F(S)$. Therefore, $fu = Sfu$ which gives that fu is another fixed point of S, which is also true for T. By uniqueness of fixed of S, $fu = u$. Hence, $u = fu = Su = Tu$, u is unique common fixed point of f, S and T in X.

Theorem 3.2 *Let (X, d) be a nonempty cone metric space and let f, T, S be three continuous self mapping on a complete cone metric space (X, d). Assume f is an injective mapping and K is a normal cone with normal constant M. Suppose f, T, S satisfy*

$$d(fTx, fSy) \le \lambda d(fx, fy) + \mu[d(fTx, fy) + d(fSy, fx)] \quad (15)$$

where $\lambda + 2\mu < 1$, for $\lambda, \mu \in [0,1)$ and $x, y \in X$. Then S and T have a unique common fixed point in X.

Proof

If we choose $\alpha = \lambda$, $\beta = 0$ and $\gamma = \mu$ in theorem 3.1, then the proof of theorem 3.2 is immediate.

Remark 3.3 *Theorem 3.2 is equivalent to the theorem 3.1 of Öztürk, M. and Metin, B. (2011).*

Theorem 3.4 *Let (X, d) be a nonempty cone metric space and let f, T, S be three continuous self mapping on a complete cone metric space (X, d). Assume f is an injective mapping and K is a normal cone with normal constant M. Suppose f, T, S satisfy*

$$d(fTx, fSy) \le \alpha_1 d(fx, fy) + \alpha_2 d(fTx, fx) + \alpha_3 d(fSy, fy) + \alpha_4 d(fTx, fy) + \alpha_5 d(fSy, fx) \quad (16)$$

where $\sum_{i=1}^{5} \alpha_i < 1$, for $\alpha_i \in [0,1)$, $i = 1,2,3,4,5.$ and $x, y \in X$. Then S and T have a unique common fixed point in X.

Proof

Let $\alpha = \alpha_1$, $\beta = \sqrt{\alpha_2 \alpha_3}$ and $\gamma = \sqrt{\alpha_4 \alpha_5}$ in the theorem 3.1 above, then

$$\alpha + 2\beta + 2\gamma = \alpha_1 + 2\sqrt{\alpha_2 \alpha_3} + 2\sqrt{\alpha_4 \alpha_5}$$

So,

$$\alpha_1 + 2\sqrt{\alpha_2 \alpha_3} + 2\sqrt{\alpha_4 \alpha_5} \le \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$$

Using this inequality in theorem 3.1, the proof of theorem 3.4 follows.

Remark 3.5 *Theorem 3.4 is the same as theorem 3.1 of Dubey et al (2014).*

Theorem 3.6 *Let* (X,d) *be a complete cone metric space and let* f,T *be two continuous self mappings of* X. *Assume* f *is an injective mapping and* K *is a normal cone with normal constant* M. *If the mapping* f,T *satisfy*

$$d(fTx, fTy) \le \alpha d(fx, fy) + \beta[d(fTx, fx) + d(fTy, fy)] + \gamma[d(fTx, fy) + d(fTy, fx)] \quad (17)$$

where $\alpha + 2\beta + 2\gamma < 1$, for $\alpha, \beta, \gamma \in [0,1)$ and $x, y \in X$. Then f and T have a unique common fixed point in X.

Proof

If we set $S = T$ in theorem 3.1, the proof of this theorem is immediate.

Corollary 3.7 *Let* f,T *be continuous self mappings of a complete cone metric space* (X,d). *Assume* f *is an injective mapping and* K *is a normal cone with normal constant* M. *If the mapping* f,T *satisfy*

$$d(fTx, fTy) \le \alpha_1 d(fx, fy) + \alpha_2 d(fTx, fx) + \alpha_3 d(fTy, fy) + \alpha_4 d(fTx, fy) + \alpha_5 d(fTy, fx) \quad (18)$$

for all $x, y \in X$ and $\sum_{i=1}^{5} \alpha_i < 1$, for $\alpha_i \in [0,1)$, $i = 1,2,\ldots,5$. Then f and T have a unique common fixed point in X.

Proof

The proof of this corollary follows from theorem 3.4 and theorem 3.6.

Corollary 3.8 *Let* T *and* S *be self mappings of a complete cone metric space* (X,d) *which satisfy*

$$d(Tx, Sy) \le \alpha d(x, y) + \beta[d(Tx, x) + d(Sy, y)] + \gamma[d(Tx, y) + d(Sy, x)] \quad (19)$$

for all $x, y \in X$ and $\alpha + 2\beta + 2\gamma < 1$, for $\alpha, \beta, \gamma \in [0,1)$. Then the pair (T,S) have a unique common fixed point in X.

Proof

The proof follows by letting $fx = x$ for all $x \in X$ in theorem 3.1.

Corollary 3.9 *Let* f, T *and* S *be three self mappings of a complete cone metric space* (X,d) *which satisfy (4).*

Then, T and S have a unique common fixed point in X.

Proof

Let $\alpha = k$, $\beta = \gamma = 0$ in theorem 3.1, then the proof of this corollary is obvious.

Corollary 3.10 *Let* (X,d) *be a nonempty cone metric space and let* f,T,S *be three continuous self mapping on a complete cone metric space* (X,d). *Assume* f *is an injective mapping and* K *is a normal cone with normal constant* M. *Suppose* f,T,S *satisfy*

$$d(fTx, fSy) \le \alpha_1 d(fx, fy) + \alpha_2 d(fTx, fx) + \alpha_3 d(fSy, fy) \quad (20)$$

where $\sum_{i=1}^{3} \alpha_i < 1$, for $\alpha_i \in [0,1)$, $i = 1,2,3.$ and $x, y \in X$. Then S and T have a unique common fixed point in X.

Proof

The proof follows by letting $\alpha_4 = \alpha_5 = 0$ in theorem 3.4.

4. Conclusion

In conclusion, we observe that our results generalized the previous results of Özturk, M. and Metin, B. (2011) and Dubey et al (2014) which could be deduced from our main results.

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