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The Extended Xlindley Distribution: Its Properties and Application to Lifetime Data

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This study introduced an Extended Xlindley Distribution (*EXD*), which was derived from the two-parameter Quasi XLindley distribution through the introduction of another parameter β. Some of its mathematical properties were obtained. The moment and maximum likelihood estimation method were used in estimating the parameters of the proposed distribution. The Kolmogorov-Smirnov goodness-of-fit test was applied to two retrieved datasets, compared against other existing models, and demonstrated greater efficiency than the considered models.

Keywords: Quasi Xlindley distribution; weighted distribution; maximum likelihood estimation; moments; kolmogorov-smirnov.

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1 Introduction

The simulation of actual events and natural phenomena using probability distributions is one of the fundamental techniques in the study of statistics and probability. Notwithstanding the fact that current probability distributions are insufficient to accurately represent the data generated by natural events, researchers have focused on developing probability distributions because of these characteristics. This influences the generalization and extension of probability distributions in various ways.

A new type of probability distribution arose due to the widespread availability of additional components. Merovci et al. (2014). In the statistical literature, many lifespan distributions have been established to provide data modelling in these applied disciplines more flexibility (Beghriche et al., 2023). Lindley Distribution was introduced by Lindley (1958) to analyse failure time data, especially in applications modelling stress - strength reliability.

The probability Distribution Function of Lindley distribution of random variable X, with scale parameter θ is given by:

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; \qquad x>0,\ \theta>0 \tag{1.1}$$

When hazard rate is unimodal or bathtub shaped then Lindley Distribution is better than Exponential Distribution (Bakouch et al., 2012). Shanker (2015) have comparative study on modeling of lifetime data using one parameter Lindley (1958) distribution and exponential distribution and concluded that there are many lifetime data where exponential distribution gives better fit than Lindley distribution. The findings led to the introduction and proposal of the Two-Parameter XLindley Distribution by Ibrahim, Shah, and Haq (2023) named Quasi-XLindley (QXL) distribution. The PDF of the continuous random variable X of follows QXL is given by:

$$f(x) = \frac{\theta}{1+\alpha}\left(\alpha + \frac{\theta(1+x)}{1+\theta}\right)e^{-\theta x}; \qquad x>0;\alpha,\theta>0 \tag{1.2}$$

with scale parameter α and θ.

Therefore, the idea of this work is to propose a new distribution called An Extended XLindley Distribution (EXD) by introducing the third (weight) parameter, with the hope that it will attract many applications in different disciplines such as reliability, survival analysis, biology and others. On applying the weighted version, the third parameter indexed, to this distribution, it is expected to be more flexible to describe different lifetime data than its sub-models.

In this paper, a three - parameter weighted quasi XLindley distribution which includes Lindley (1958) distribution, Exponential Lindley distribution and quasi XLindley distribution as particular cases, has been proposed and discussed. Its moments about origin and central moments, coefficient of variation, skewness, kurtosis and index of dispersion have been derived (Akaike, 1974; Meriem et al., 2022). The hazard rate function and Survival function of the distribution have been derived and their shapes have been discussed for varying values of the parameters. The estimation of its parameters has been discussed using maximum likelihood method. Finally, the goodness of fit and the applications of the distribution have been explained through two failure time data and the fit has been compared with one parameter Lindley distribution and the Quasi XLindley distribution.

2 Methods

2.1 Weighted Quasi XLindley Distribution

This paper outlines the methods and principles applied in conducting this research which is Weighted Quasi Xlindley distribution with scale parameter α and shape parameter θ and β. The method of moment, the coefficient of variation, Coefficient of Skewness, Coefficient of Kurtosis, Index of Dispersion, the survival function, the hazard function, and maximum likelihood

estimation were considered by the study. This study employed the incorporation of parameter into the existing distributions. The pdf of Quasi XLindley distribution for a random variable X with scale parameter α and θ is defined as

$$f(x) = \frac{\theta}{1+\alpha}\left(\alpha + \frac{\theta(1+x)}{1+\theta})e^{-\theta x}\right); \quad \alpha,\theta > 0; x > 0 \tag{2.1}$$

Thus, the probability density function of Weighted Quasi Xlindley distribution (WQXL) is obtained given as:

$$f_w(x) = \frac{w(x)f(x)}{E[w(x)]}; \quad x > 0, \tag{2.2}$$

where

$E[w(x)] = \int_0^\infty w(x)f(x)dx$ is the normalizing factor obtained to make the total probability equal to unity

and

$w(x) = x^\beta$ (as suggested by Rather and Ozel (2020).

It represents the weighted version of x , known as the size-biased version, and the distribution is referred to as the size-biased distribution (Patil, Rao, Zelen, & Patil, 1987) with the pdf given in equation (4).

Given that expectation of a continuous random variable X, denoted as $\mathbb{E}[X]$, is defined as:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

where $f_X(x)$ is the probability density function (PDF) of X.

$$\therefore E(x^\beta) = \int_0^\infty x^\beta f(x)dx. \tag{2.3}$$

substituting equation (3) and (5) in (4), then solve,

$$f_w(x) = \frac{\theta^{(1+\beta)}}{1+\alpha+\beta}\frac{x^\beta(\alpha+\theta+x\theta+\alpha\theta)}{\Gamma(1+\beta)}e^{-x\theta}. \tag{2.4}$$

α and θ are the existing scale while β is the new introduced parameter.

Proposition 1: Suppose a random variable X follows a Weighted Quasi XLindley Distribution, that is, $X \mid \alpha,\theta,\beta \sim WQXL(\alpha,\theta,\beta)$ then the pdf of random variable X is given as:

$$f(x;\alpha,\theta,\beta) = \frac{\theta^{(1+\beta)}}{1+\alpha+\beta}\frac{x^\beta(\alpha+\theta+x\theta+\alpha\theta)}{\Gamma(1+\beta)}e^{-x\theta}; x > 0, \alpha,\theta,\beta > 0 \tag{2.5}$$

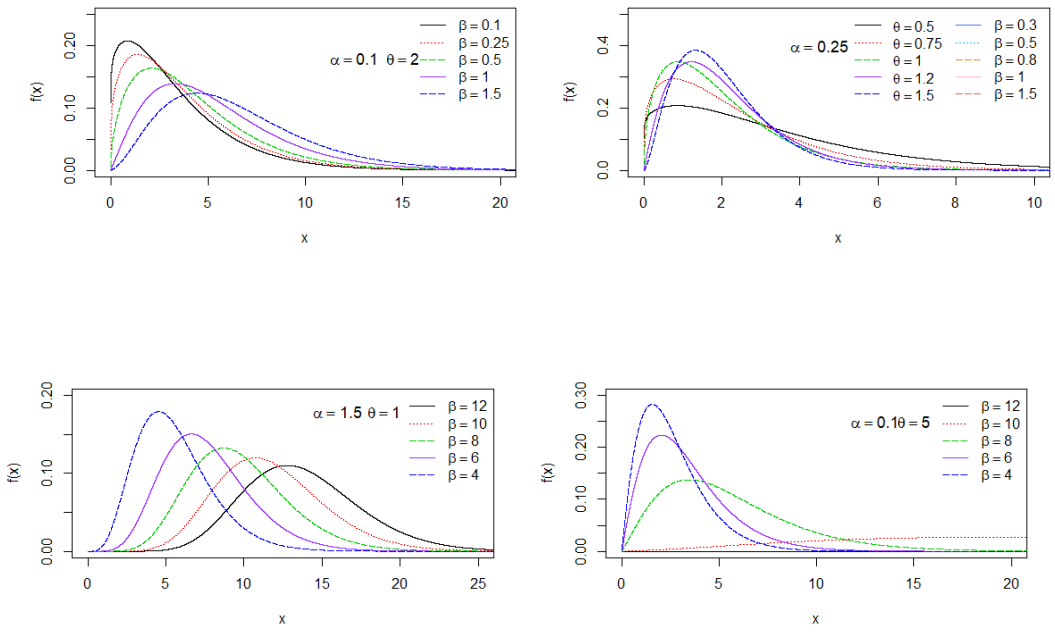


Fig. 1. The probability density function of WQXLD for various choices of parameters

The graphs presented in Fig. 1 shows the pattern of the pdf of the proposed WQXL distribution at various values of α, θ and β . The pdf shows different shapes which makes it flexible to capture different shapes as may be exhibited by different data fields.

2.2 Moment

The moments of distribution are used to describe the characteristics or shape of the distribution. The r^{th} moment of a continuous random variable x:

$$E(X^r) = \mu'_r = \int_0^\infty x^r f(x)dx \tag{2.6}$$

where 'r' is a positive integer. Hence, the r^{th} moment of a random variable x with the WQXL distribution is obtained as:

$$E(X^r) = \mu'_r = \frac{\theta^{-r}(1+r+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+r+\beta)}{1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta)}. \tag{2.7}$$

The first four moments of WQXLD are:

$$\mu'_1 = \frac{(1+\beta)(2+\alpha+\beta+\theta+\alpha\theta)}{\theta(1+\alpha+\beta+\theta+\alpha\theta)}. \tag{2.8}$$

$$\mu'_2 = \frac{(1+\beta)(2+\beta)(3+\alpha+\beta+\theta+\alpha\theta)}{\theta^2(1+\alpha+\beta+\theta+\alpha\theta)}. \tag{2.9}$$

$$\mu'_3 = \frac{(4+\alpha+\beta+\theta+\alpha\theta)\Gamma(4+\beta)}{\theta^3(1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta)}. \tag{2.10}$$

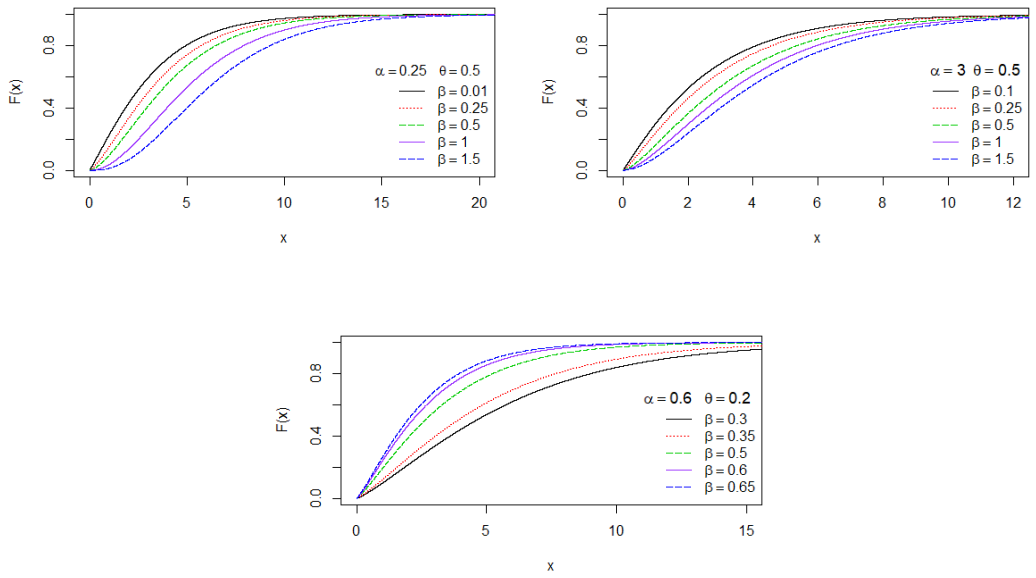


Fig. 2. The cumulative distribution function of WQXL distribution for various choices of parameters

$$\mu_4' = \frac{(5+\alpha+\beta+\theta+\alpha\theta)\Gamma(5+\beta)}{\theta^4(1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta)}. \tag{2.11}$$

Given the random variable x having the WQXLD with parameters α, θ and β the variance, coefficient of variation, skewness, kurtosis and index of variation is given by

$$\sigma^2 = \frac{(1+\beta)(2+\beta^2+4\theta+\theta^2+\alpha^2(1+\theta)^2+2\alpha(1+\theta)(2+\beta+\theta)+\beta(3+2\theta)}{\theta^2(1+\alpha+\beta+\theta+\alpha\theta)^2}. \tag{2.12}$$

$$CV = \frac{\sqrt{2+\beta^2+4\theta+\theta^2+\alpha^2(1+\theta)^2+2\alpha(1+\theta)(2+\beta+\theta)+\beta(3+2\theta)}}{\theta(1+\alpha+\beta+\theta+\alpha\theta)(2+\alpha+\beta+\theta+\alpha\theta)}. \tag{2.13}$$

$$S_k = \frac{(4+\alpha+\beta+\theta+\alpha\theta)\Gamma(4+\beta)}{\theta^3(1+\alpha+\beta+\theta+\alpha\theta)\left[\frac{(1+\beta)(2+\beta^2+4\theta+\theta^2+\alpha^2(1+\theta)^2+2\alpha(1+\theta)(2+\beta+\theta)+\beta)(3+2\theta)}{\theta^2(1+\alpha+\beta+\theta+\alpha\theta)^2}\right]^{\frac{3}{2}}\Gamma(1+\beta)}. \tag{2.14}$$

$$K_s = \frac{(1+\alpha+\beta+\theta+\alpha\theta)^3(5+\alpha+\beta+theta+\alpha\theta)\Gamma(5+\beta)}{(1+\beta)^2(2+\beta^2+4\theta+\theta^2+\alpha^2(1+\theta)^2+2\alpha(1+theta))(2+\beta+\theta)+\beta(3+2\theta))^2\Gamma(1+\beta)}. \tag{2.15}$$

$$ID_s = \frac{(2+\beta^2+4\theta+\theta^2+\alpha^2(1+\theta)(2+\beta+\theta)+\beta(3+2\theta))}{\theta(1+\alpha+\beta+\theta+\alpha\theta)(2+\alpha+\beta+\theta+\alpha\theta)}. \tag{2.16}$$

Table 1. Statistical Metrics for α = 0.5 and 1.0

Parameters			Statistic				
α	θ	β	μ	σ^2	Sk	Ks	ID
0.5	0.5	0.5	3.81818	15.42149	3.24635	7.66329	4.03896
		1.0	3.84615	29.51479	2.30271	3.99484	7.67385
		1.2	3.85507	35.74479	2.26331	3.41528	9.27215
		2.0	3.88235	63.98615	3.02181	2.32132	16.48128
	1.0	0.5	1.71428	4.02551	2.71879	6.07461	2.34821
		1.0	1.75000	7.43750	2.07066	3.47094	4.25000
		1.2	1.76190	8.96426	2.05592	3.01722	5.08782
		2.0	1.80000	15.96000	2.73753	2.11996	8.86667
	1.2	0.5	1.38157	2.81985	2.59035	5.70205	2.04104
		1.0	1.414729	5.169085	2.00632	3.32899	3.65376
		1.2	1.425926	6.223525	1.99674	2.90575	4.36455
		2.0	1.462264	11.06933	2.65266	2.05891	7.56999
	2.0	0.5	0.75000	1.03125	2.27313	4.80610	1.37500
		1.0	0.77272	1.85743	1.83150	2.95025	2.40374
		1.2	0.78070	2.23120	1.83224	2.60055	2.85795
		2.0	0.80769	3.96301	2.40832	1.88031	4.90659
1.0	0.5	0.5	3.42857	16.10204	2.71879	6.07461	4.69643
		1.0	3.50000	29.75000	2.07066	3.47094	8.50000
		1.2	3.52381	35.85705	2.05592	3.01722	10.17565
		2.0	3.60000	63.84000	2.73753	2.11996	17.73333
	1.0	0.5	1.55555	3.83024	2.65323	6.22178	2.46230
		1.0	1.60000	7.04000	2.05576	3.63184	4.40000
		1.2	1.61538	8.49207	2.03608	3.16369	5.25700
		2.0	1.66666	15.22222	2.59782	2.21947	9.13333
	1.2	0.5	1.25850	2.61472	2.64037	6.28454	2.07764
		1.0	1.29629	4.80109	2.05373	3.68742	3.70370
		1.2	1.30952	5.79308	2.03215	3.21372	4.42381
		2.0	1.35416	10.40581	2.55881	2.25334	7.68429
	2.0	0.5	0.69230	0.89090	2.61876	6.51118	1.28686
		1.0	0.71428	1.63265	2.05439	3.86880	2.28571
		1.2	0.72222	1.97172	2.02638	3.37610	2.73008
		2.0	0.75000	3.56250	2.45387	2.36380	4.75000

Table 2. Statistical Metrics for α = 1.2 and 2.0

Parameters			Statistic				
α	θ	β	μ	σ^2	Sk	Ks	ID
1.2	0.5	0.5	3.31578	16.24238	2.59035	5.70205	4.89850
		1.0	3.39534	29.77393	2.00632	3.32899	8.76904
		1.2	3.42222	35.84750	1.99674	2.90575	10.47492
		2.0	3.50943	63.75934	2.65266	2.05891	18.16799
	1.0	0.5	1.51020	3.76520	2.64037	6.28454	2.49317
		1.0	1.55555	6.91358	2.05373	3.68742	4.44444
		1.2	1.57142	8.34204	2.03215	3.21372	5.30857
		2.0	1.62500	14.98437	2.55881	2.25334	9.22115
	1.2	0.5	1.22347	2.55079	2.66223	6.48996	2.08488
		1.0	1.26141	4.68737	2.07163	3.81140	3.71596
		1.2	1.27483	5.65956	2.04609	3.32024	4.43945
		2.0	1.32066	10.19542	2.53510	2.32114	7.71993
	2.0	0.5	0.67605	0.85016	2.74406	7.17596	1.25754
		1.0	0.69736	1.56630	2.13451	4.22168	2.24603
		1.2	0.70512	1.89458	2.09651	3.67332	2.68687
		2.0	0.73255	3.44010	2.47913	2.54849	4.69601
2.0	0.5	0.5	3.00000	16.50000	2.27313	4.80610	5.50000
		1.0	3.09090	29.71900	1.83150	2.95025	9.61497
		1.2	3.12280	35.69930	1.83224	2.60055	11.43180
		2.0	3.23076	63.40828	2.40832	1.88031	19.62637
	1.0	0.5	1.38461	3.56360	2.61876	6.51118	2.57372
		1.0	1.42857	6.53061	2.05439	3.86880	4.57143
		1.2	1.44444	7.88691	2.02638	3.37610	5.46017
		2.0	1.50000	14.25000	2.45387	2.36380	9.50000
	1.2	0.5	1.12676	2.36157	2.74406	7.17596	2.09590
		1.0	1.16228	4.35085	2.13451	4.22168	3.74338
		1.2	1.17521	5.26274	2.09651	3.67332	4.47812
		2.0	1.22093	9.55584	2.47913	2.54849	7.82669
	2.0	0.5	0.63157	0.73597	3.18247	9.68443	1.16530
		1.0	0.65000	1.37750	2.41228	5.53358	2.11923
		1.2	0.65686	1.67362	2.34056	4.77612	2.54791
		2.0	0.68181	3.08057	2.58928	3.23308	4.51818

Tables 1 and 2 show the nature of the mean (μ_3), variance (σ^2), coefficient of skewness (S_k), coefficient of kurtosis (Ks) and index of dispersion (ID) of the WQXLD for varying values of the parameters. The coefficient of skewness (S_k) indicates a positively skewed distribution that is moderately to highly asymmetric and coefficient of kurtosis (K_s) indicates a range of tail behaviors from slightly platykurtic (1.88031) to highly leptokurtic (9.68443).

2.3 Maximum Likelihood Estimation (MLEs)

The Maximum Likelihood Estimate (MLE) is a widely used method for estimating the parameters of an assumed probability distribution. This is because of MLE estimators have desirable properties such as consistency, asymptotic efficiency, and invariance. To obtain the maximum likelihood estimators of the parameters of the WQXLD, let x_1,x_2,...,x_n be a random sample of size n from the WQXLD with the log-likelihood function is defined as

$$L(\alpha,\theta,\beta;x_i)=\prod_{i=1}^{n} f(x_i;\alpha,\theta,\beta,) \tag{2.17}$$

This implies that

$$L(\alpha,\theta,\beta;x_i)=\left(\frac{\theta^{(1+\beta)}}{1+\alpha+\beta}\right)\prod_{i=1}^{n}\frac{x_i^{\beta}(\alpha+\theta+x\theta+\alpha\theta)}{\Gamma(1+\beta)}e^{-n\theta\bar{x}} \tag{2.18}$$

$$ln\, L(\alpha,\theta,\beta;x_i)=ln\left(\frac{\theta^{(1+\beta)}}{1+\alpha+\beta}\right)\sum_{i=1}^{n} ln\left(\frac{x_i^{\beta}(\alpha+\theta+x\theta+\alpha\theta)}{\Gamma(1+\beta)}e^{-n\theta\bar{x}}\right) \tag{2.19}$$

$$\ell(\alpha,\theta,\beta,x_i)=(1+\beta)ln(\theta)-ln(1+\alpha+\beta)+\beta\sum_{i=1}^{n} ln(x_i)+g(\alpha,\theta,\beta,x_i) \tag{2.20}$$

where,

$$g(\alpha,\theta,\beta,x_i)=ln(x_i)+ln(\alpha+\theta+x\theta+\alpha\theta)-ln(\Gamma(1+\beta))-n\theta$$

The maximum likelihood estimates (MLEs) $\hat{\alpha}$, $\hat{\theta}$, and $\hat{\beta}$ are derived by differentiating the log-likelihood function with respect to the parameters α, θ, and β, respectively. Setting these partial derivatives to zero results in a system of non-linear equations, the solutions of which provide the parameter estimates.
For θ,

$$\frac{\partial\ell}{\partial\theta}=\sum_{i=1}^{n}\left(\frac{1+\beta}{\theta}+\frac{1+x_i+\alpha}{\alpha+\theta+x_i\theta+\alpha\theta}-n\right).$$

For α,

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{1+\alpha+\beta}+\sum_{i=1}^{n}\frac{1}{\alpha+\theta+x_i\theta+\alpha\theta}$$

For β,

$$\frac{\partial\ell}{\partial\beta}=n\ln(\theta)-\psi(1+\beta)+\sum_{i=1}^{n}\ln(x_i)$$

These three natural log likelihood equations cannot be solved directly. However, the Fisher's scoring method was used to solve these equations and an Hessian Matrix of the log-likelihood function *ln L*, which consists of second-order partial derivatives of *ln L* with respect to the parameters α,θ, β. was obtained

$$I(\alpha,\theta,\beta)=\begin{bmatrix}\frac{\partial^2 lnL}{\partial\alpha^2} & \frac{\partial^2 lnL}{\partial\alpha\partial\theta} & \frac{\partial^2 lnL}{\partial\alpha\partial\beta}\\ \frac{\partial^2 lnL}{\partial\theta\partial\alpha} & \frac{\partial^2 lnL}{\partial\theta^2} & \frac{\partial^2 lnL}{\partial\theta\partial\beta}\\ \frac{\partial^2 lnL}{\partial\beta\partial\alpha} & \frac{\partial^2 lnL}{\partial\beta\partial\theta} & \frac{\partial^2 lnL}{\partial\beta^2}\end{bmatrix}$$

where α_0, θ_0 and β_0 are the initial values of α, θ and β respectively. These equations were solved iteratively till sufficiently close values of $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\beta}$ are obtained.

2.4 Reliability analysis

Survival Function

The survival function is a function that gives the probability that a patient, device, or other object of interest will survive past certain time (Jiang & Guterman, 2024). The survival function is the complementary cumulative distribution of the lifetime. Let the lifetime x be a continuous random variable with cumulative density function $F(x)$ and probability density function $f(x)$ on the interval $[0, \infty]$. its survival function is

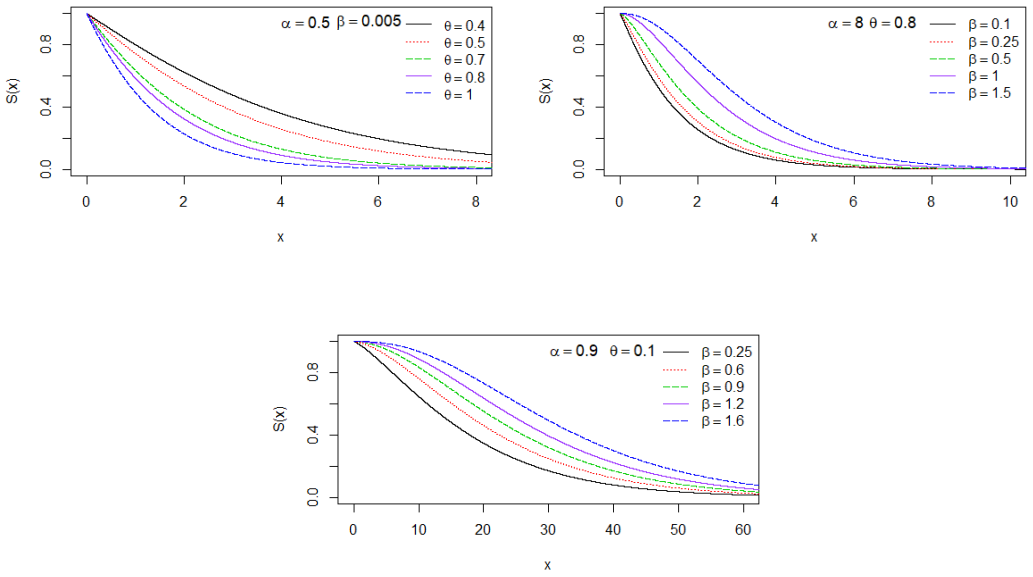
$$S(X) = 1 - F(x)$$

Therefore, the survival function for WQXLD is given as:

$$S(x,\alpha,\theta,\beta) = 1 - F(x,\alpha,\theta,\beta)$$

$$S(x,\alpha,\theta,\beta) = 1 + \frac{x^{\beta}\theta^{\beta}(x\theta)^{-\beta}\left(-(1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta) + (\alpha+\theta+\alpha\theta)\Gamma(1+\beta,x\theta) + \Gamma(2+\beta,x\theta)\right)}{(1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta)}.$$

Fig. 3 presented the shapes of the survival function of the proposed WQXLD at selected values of α, θ and β. The plots also show a decreasing rate of survival at time x.



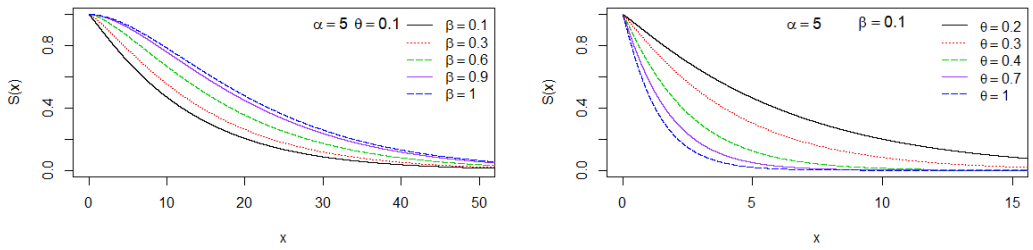


Fig. 3. The survival function of WQXLD for various choices of parameter

Hazard Function

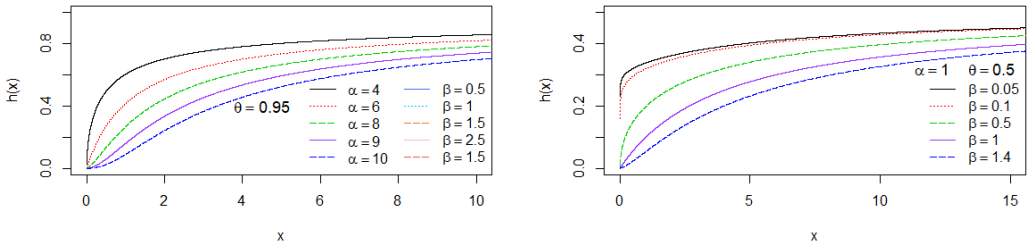
The hazard function $h(x)$ of an event is the probability of the failure of the event at time x. It is the probability that an event will fail at a given time x. The hazard function $h(x)$ is given as:

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{s(x)} \tag{2.21}$$

the hazard function of WQXLD is given as

$$h(x,\alpha,\theta,\beta) = \frac{e^{-x\theta}\theta\left(\alpha+\frac{(1+x)\theta}{1+\theta}\right)}{(1+\alpha)\left(1+\frac{x^{\beta}\theta^{\beta}(x\theta)^{-\beta}(-(1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta)+(\alpha+\theta+\alpha\theta)\Gamma(1+\beta,x\theta)+\Gamma(2+\beta,x\theta))}{(1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta)}\right)} \tag{2.22}$$

Fig. 4 presented the shapes of the hazard function of the proposed WQXLD at selected values of α, θ and β. The plots show an increasing pattern which suggests that items are more likely to fail at the beginning after which they maintain a constant failure rate. This means that the failure may likely happen during the useful life of the item and failure occurs at random.



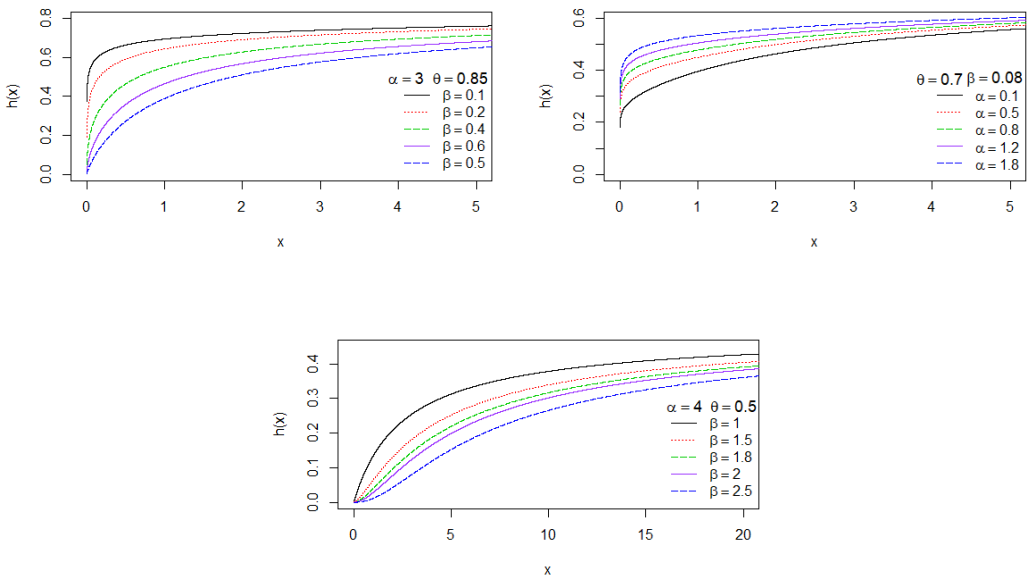


Fig. 4. The hazard function of WQXLD for various choices of parameter

2.5 Simulation study

In this section, we evaluate $\hat{\alpha}_{MLE}$, $\hat{\theta}_{MLE}$ and $\hat{\beta}_{MLE}$ through a brief simulation study. The simulation study of the WQXLD is carried out by choosing random samples, say n = 50, 100, ..., 1000. These samples are obtained using the inverse cdf. The simulation study is conducted for the parameter combinations α, θ and β respectively. The evaluation of the performance of $\hat{\alpha}_{MLE}$, $\hat{\theta}_{MLE}$ and $\hat{\beta}_{MLE}$ is based on two criteria. These criteria are given by

Mean Square Error (MSE)
MSE $(\hat{\alpha}_{MLE}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2$
Bias
Bias $(\hat{\alpha}_{MLE}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)$

The above evaluation criteria are also computed for $\hat{\theta}_{MLE}$ and $\hat{\beta}_{MLE}$. The simulation study is performed using the optim()R-function with argument method = "L-BFGSB". The simulation results are presented numerically in Tables 3.

Interpretation

From the results of the simulation of WQXLD in Table 3, the followings were observed:

1. As n increases (i.e. as $n \to \infty$), the values of $\hat{\alpha}_{MLE}$, $\hat{\theta}_{MLE}$ and $\hat{\beta}_{MLE}$ become more consistent and closer to their true values
2. As n increases (i.e. as $n \to \infty$), MSE of $\hat{\alpha}_{MLE}$, $\hat{\theta}_{MLE}$ and $\hat{\beta}_{MLE}$ tends to 0, which is expected since larger samples reduce the variance of the estimator.
3. As n increases (i.e. as $n \to \infty$), Biases of $\hat{\alpha}_{MLE}$, $\hat{\theta}_{MLE}$ and $\hat{\beta}_{MLE}$ tends towards to 0, suggesting that the estimators are asymptotically unbiased.

Table 3. The numerical illustration of the Simulation Study of the WQXL distribution for $\alpha = 1.0$, $\theta = 2.0$ and $\beta = 1.0$

n	Parameter	MLEs	MSEs	Biases
50	α	0.00004	0.00000	0.00004
	θ	0.87713	0.45851	0.67713
	β	0.58121	0.23157	0.48121
100	α	0.00068	0.00469	0.00006
	θ	0.74453	0.29650	0.54453
	β	0.04749	0.14424	0.37979
200	α	0.00035	0.12124	0.00034
	θ	0.60109	0.16884	0.41091
	β	0.50133	0.16107	0.40133
300	α	0.00000	0.00000	0.00000
	θ	0.58875	0.15113	0.38875
	β	0.42417	0.10508	0.32417
500	α	0.00014	0.16154	0.00000
	θ	0.58625	0.14919	0.38625
	β	0.42961	0.10865	0.32961
700	α	0.00000	0.00000	0.37062
	θ	0.17765	0.17765	0.42148
	β	0.11429	0.10428	0.33807
1000	α	0.00000	0.00000	0.00004
	θ	0.65337	0.18994	0.04380
	β	0.45601	0.10184	0.25615

2.6 Models diagnostic method

The model selection criteria considered in this thesis are the AIC (Akaike Information Criterion) by Akaike (1974), AICC (Corrected Akaike Information Criterion) by Kletting and Glatting (2009), HQIC (Hannan-Quinn Information Criterion) by Maïnassara and Kokonendji (2016), BIC (Bayesian Information Criterion) by Weakliem (1999) and Kolmogorov-Sminorv (1948). Where the AIC, AICC, HQIC, BIC and K-S are obtained as follows:

$$AIC = 2k - 2ln(L) \tag{2.23}$$

$$AICC = AIC + \frac{k(k-1)}{n-k-1} \tag{2.24}$$

$$HQIC = -2\ln(L) + 6\ln(\ln(n)) \tag{2.25}$$

$$BIC = kln(n) - 2ln(L) \tag{2.26}$$

$$K.S = sup_x|F_n(x) - F_0(x)| \tag{2.27}$$

Where k is the number of parameters in the distribution and n is the number of observations.

3 Applications

Dataset 1

The dataset represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported by Lee and Wang (2003)

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52,4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80,25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32,10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90,2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66,

1.05, 2.69, 4.23, 5.41,7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12,1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62,7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46,4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02,2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07,21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Dataset 2

The following data set represents the tree circumferences in Marshall, Minnesota and reported by Shakil et al. (2010)
1.8, 1.8, 1.9, 2.4, 3.1, 3.4, 3.7, 3.7, 3.8, 3.9, 4.0, 4.1, 4.9, 5.1, 5.1, 5.2, 5.3, 5.5, 8.3, 13.7

Table 4. Parameter estimates and goodness of fit test statistics for dataset 1

Dataset	Model	MLE	SE	-2logL	AIC	AICC	HQIC	BIC	K-S	KS p-value
1	**WQXLD**	$\hat{\alpha} = 8.00881$ $\hat{\theta} = 0.17517$ $\hat{\beta} = 0.36757$	$\hat{\alpha} = 14.08806$ $\hat{\theta} = 0.022891$ $\hat{\beta} = 0.189713$	**791.4412**	**797.4412**	**797.6348**	**800.9176**	**801.1453**	**0.5544**	**0.00000**
	QXLD	$\hat{\alpha} = 9.36563$ $\hat{\theta} = 0.11591$	$\hat{\alpha} = 17.612729$ $\hat{\theta} = 0.018359$	828.8488	832.8488	832.9448	835.1664	838.5529	0.86118	0.00000
	ATPLD	$\hat{\alpha} = 9.47260$ $\hat{\theta} = 0.10677$ $\hat{\beta} = 0.00000$	$\hat{\alpha} = 1.2380$ $\hat{\theta} = 2.95889$ $\hat{\beta} = 2.64226$	828.6838	834.6838	834.8774	838.1602	838.3879	0.6704	0.00022
	MQLD	$\hat{\alpha} = 0.17340$ $\hat{\theta} = 4.28457$ $\hat{\beta} = 7.2913652$	$\hat{\alpha} = 2.18535$ $\hat{\theta} = 0.11520$ $\hat{\beta} = 4.15722$	826.1754	832.1754	832.3689	835.6518	835.8794	0.97525	0.00000
	WQLD	$\hat{\alpha} = 9.36616$ $\hat{\theta} = 0.13520$ $\hat{\beta} = 1.15880$	$\hat{\alpha} = 10.57737$ $\hat{\theta} = 0.02030$ $\hat{\beta} = 0.13066$	827.1847	831.1847	831.2807	833.5023	836.8888	0.82186	0.00000
2	**WQXLD**	$\hat{\alpha} = 1570.20939$ $\hat{\theta} = 2.24865$ $\hat{\beta} = 8.50471$	$\hat{\alpha} = 0.00033$ $\hat{\theta} = 5.10547$ $\hat{\beta} = 2.15390$	**48.51669**	**54.51669**	**55.92846**	**55.19676**	**54.60574**	**0.5034**	**0.00000**
	QXLD	$\hat{\alpha} = 0.0010$ $\hat{\theta} = 0.158$	$\hat{\alpha} = 1.334$ $\hat{\theta} = 0.37917$	94.15925	98.07925	98.77513	98.462	100.07	0.95628	0.00000
	ATPLD	$\hat{\alpha} = 0.0100$ $\hat{\theta} = 0.92572$ $\hat{\beta} = 4.19818$	$\hat{\alpha} = 1.11023$ $\hat{\theta} = 0.29943$ $\hat{\beta} = 1.27832$	85.97328	91.1676	92.6676	91.75073	94.15479	0.74633	0.42150
	MQLD	$\hat{\alpha} = 0.00110$ $\hat{\theta} = 0.92556$ $\hat{\beta} = 4.19745$	$\hat{\alpha} = 3.34586$ $\hat{\theta} = 0.29938$ $\hat{\beta} = 1.27809$	85.16757	91.16757	92.66757	91.7507	91.15903	0.55395	0.00000
	WQLD	$\hat{\alpha} = 0.10000$ $\hat{\theta} = 0.92692$ $\hat{\beta} = 3.20659$	$\hat{\alpha} = 8.54498$ $\hat{\theta} = 0.05449$ $\hat{\beta} = 1.54498$	91.17018	91.17018	92.67018	91.75332	94.15738	0.82349	0.33151

Table 4 shows the goodness of fit tested with the retrieved datasets of 128 patients with bladder cancer to recovery and 20 Tree circumferences in Marshall, Minnesota. It was observed that the WQXLD produced the smallest values in terms of log likelihood values, AIC, AICC, HQIC, BIC and K-S compared to other competing models for both datasets. This suggests that the WQXLD model provides more accurate estimates as well as better fits among all the considered distributions which have been fitted to the patients with bladder cancer dataset.

Additionally, the p-values associated with the KS test indicate the statistical significance of the fit. For this analysis, most p-values are close to zero, which might reflect specific characteristics of the datasets or the limitations of certain models in capturing the data's distribution.

4 Conclusion

This paper derived a new distribution, Weighted Quasi XLindley Distribution (*WQXLD*) by introducing another parameter β. The parameter is the weighted version of x called the size-biased version of x. The mathematical properties such as the method

of moment, maximum likelihood estimate was used to estimate the parameters. Coefficient of variation, the survival function, the hazard function were discussed and Finally, two datasets were used to check real life applicability of the WQXLD. The negative log-likelihood, criterion values such as *AIC*, *AICC*, *HQIC* and *BIC* were used for comparison with *QXL*, *ATPLD*, *WQLD*, and *MQLD*. The results from Table 4 reveal that *WQXLD* produced low values in terms of *-LL*, *AIC*, *AICC*, *HQIC* and *BIC* which shows that comparatively *WQXLD* is more efficient than the existing models considered. Hence, the *WQXLD* can be considered an important life time distribution for modelling data.

5 Future Research

Having consider WQXLD as an important lifetime distribution for modeling data, it is therefore suggested that future research should include development and study of higher-order generalizations of the WQXLD by introducing additional parameters to capture even more complex data patterns. Explore hybrid models that integrate the WQXLD with other statistical distributions to improve flexibility. Extend the WQXLD to multivariate contexts, enabling the modeling of correlated data across multiple dimensions. Investigate the applicability of the WQXLD in emerging fields such as Machine Learning, Artificial Intelligence, or Financial Modeling.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Authors have declared that no competing interests exist.

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