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Research Article

Approximate Solution of Perturbed Volterra-Fredholm Integrodifferential Equations by Chebyshev-Galerkin Method

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In this work, we obtain the approximate solution for the integrodifferential equations by adding perturbation terms to the right hand side of integrodifferential equation and then solve the resulting equation using Chebyshev-Galerkin method. Details of the method are presented and some numerical results along with absolute errors are given to clarify the method. Where necessary, we made comparison with the results obtained previously in the literature. The results obtained reveal the accuracy of the method presented in this study.

1. Introduction

Integrodifferential equations (IDEs) arise in many branches of mathematical sciences such as financial mathematics, mathematical modeling, and control theory, which are usually difficult to solve analytically, so numerical approach is required. Different methods have been used to obtain the solution of both linear and nonlinear IDEs such as Galerkin method [1–4], homotopy perturbation [5, 6], Tau method [7–9], spline collocation method [10–12], Taylor collocation [13–17], finite element approach [18–23], Legendre polynomials [24], variational iteration method [25], Haar wavelets [26], Krein's method [27], and Bessel collocation method [28], to mention a few.

The present work is motivated by the desire to obtain numerical solutions to initial value problems for integrodifferential equations via perturbed Chebyshev-Galerkin method. This paper is organized as follows. In Section 2, Chebyshev polynomial is discussed. In Section 3, preliminary steps towards application of the perturbed Chebyshev-Galerkin method are introduced. In Section 4, some numerical results are provided to demonstrate the efficiency and accuracy of using perturbed Chebyshev-Galerkin method and compared with those of [1, 29] and lastly, Section 5 is the conclusion.

2. Chebyshev Polynomial

Chebyshev polynomials are widely used in applications in mathematics, mathematical physics, engineering, and computer science. Chebyshev polynomial is an orthogonal polynomial which satisfy the recurrence relation

$$T_n(x) = \cos\left[n\cos^{-1}\left(\frac{2x-(a+b)}{b-a}\right)\right], \quad a \le x \le b,$$
$$T_{n+1}(x) = \left(\frac{2x-(a+b)}{b-a}\right)T_n(x) - T_{n-1}(x), \quad n \ge 1. \tag{1}$$

In recent years, a lot of attention has been devoted to the study of Chebyshev methods to investigate various scientific models. Using these methods made it possible to solve differential equations of different forms [1, 7, 30–32], irrespective of the order of the differential equations.

In certain applications we need expressions for the products like $T_i(x)T_j(x)$, which comes easily from

$$T_i(x)\,T_j(x) = \frac{1}{2}\left[T_{i+j}(x) + T_{i-j}(x)\right]. \tag{2}$$

3. Perturbed Chebyshev-Galerkin Method

We will consider the numerical solution of a class of linear Volterra-Fredholm integrodifferential initial value problems of the form

$$\sum_{l=0}^{m} p_l(x)\frac{d^l}{dx^l}\phi(x) = f(x) + \lambda_1\int_a^b \tau_1(x,t)\phi(t)\,dt + \lambda_2\int_a^x \tau_2(x,t)\phi(t)\,dt, \tag{3}$$

$$\phi^j(a) = \gamma_j, \quad j = 0,1,\ldots,m-1, \tag{4}$$

where $\phi(x)$ is the unknown function, $p_l(x)$, $f(x)$, and $\tau_k(x,t)$, $k = 1,2$, are known functions, m is the order of (3), and λ_k, $k = 1,2$, are real numbers. Unless otherwise stated, x will always be the independent variable of the functions which appear throughout this paper and will be defined in a finite interval $[a,b]$. Moreover suppose that $\phi_n(x)$ be the approximate solution of degree n to $\phi(x)$, so we write

$$\phi_n(x) = \sum_{i=0}^{n}\beta_i T_i\left(\frac{2x-(a+b)}{b-a}\right), \tag{5}$$

where β_i are determined by adding perturbation terms

$$H_n(x) = \sum_{l=0}^{m}\mu_{l+1}T_{n-m+l+1}\left(\frac{2x-(a+b)}{b-a}\right) \tag{6}$$

to the right hand side of (3); we obtain

$$\sum_{l=0}^{m} p_l(x)\frac{d^l}{dx^l}\phi_n(x) = f(x) + \lambda_1\int_a^b \tau_1(x,t)\phi_n(t)\,dt + \lambda_2\int_a^x \tau_2(x,t)\phi_n(t)\,dt + H_n(x). \tag{7}$$

Applying Chebyshev-Galerkin approach discussed in [1], that is, multiplying both sides of (7) by $T_s((2x-(a+b))/(b-a))$, $s = m, m+1,\ldots,n+m+1$, and then integrating the resulting equation over interval $[a,b]$, we obtain

$$\int_a^b\left[\sum_{l=0}^{m} p_l(x)\sum_{i=0}^{n}\left(\frac{2}{b-a}\right)^l\beta_i\frac{d^l}{dx^l}T_i\left(\frac{2x-(a+b)}{b-a}\right) - \lambda_1\int_a^b\tau_1(x,t)\sum_{i=0}^{n}\beta_i T_i\left(\frac{2t-(a+b)}{b-a}\right)dt - \lambda_2\int_a^x\tau_2(x,t)\sum_{i=0}^{n}\beta_i T_i\left(\frac{2t-(a+b)}{b-a}\right)dt - \sum_{l=0}^{m}\mu_{l+1}T_{n-m+l+1}\left(\frac{2x-(a+b)}{b-a}\right)\right]\cdot T_s\left(\frac{2x-(a+b)}{b-a}\right)dx = f(x)\cdot T_s\left(\frac{2x-(a+b)}{b-a}\right)dx, \quad s = m,m+1,\ldots,n+m+1. \tag{8}$$

From (8), we have

$$\theta\beta = F, \tag{9}$$

where θ is a matrix of $(n+2)\times(n+m+2)$, β and F are column matrices of $(n+2)\times 1$, and the other equations are derived from the initial conditions (4); that is,

$$\frac{d^j}{dx^j}\phi_n(a) = \sum_{i=0}^{n}\left(\frac{2}{b-a}\right)^j\beta_i\frac{d^j}{dx^j}T_i\left(\frac{2x-(a+b)}{b-a}\right)\Bigg|_{x=a} = \gamma_j, \quad j = 0,1,m-1. \tag{10}$$

Substitute the values of $\beta_0,\beta_1,\ldots,\beta_n$ obtained from (9) and (10) in (5) to obtain the approximate solution of degree n.

4. Numerical Examples

Example 1 (see [1, 33]). Consider the Fredholm integrodifferential equation

$$\phi'(x) = \phi(x) - \frac{1}{2}x + \frac{1}{1+x} - \log(1+x) + \frac{1}{(\log(2))^2}\int_0^1\frac{x}{1+t}\phi(t)\,dt, \quad 0\le x\le 1 \tag{11}$$

subject to initial condition

$$\phi(0) = 0 \text{ and whose exact solution is } \phi(x) = \log(1+x). \tag{12}$$

The absolute errors are tabulated in Table 1 at different n. Table 5 exhibits a comparison between the errors obtained by

Table 1: Absolute errors for Example 1 at different n.

x	$n = 4$	$n = 8$	$n = 12$
0	0	0	0
0.1	7.8696$e-06$	2.9221$e-08$	3.9291$e-11$
0.2	1.3565$e-04$	7.2475$e-08$	8.0337$e-12$
0.3	1.9086$e-04$	2.5687$e-08$	7.0680$e-11$
0.4	1.4192$e-04$	2.7660$e-08$	1.8044$e-11$
0.5	3.6209$e-05$	5.6989$e-08$	6.6114$e-11$
0.6	5.5319$e-05$	1.2399$e-07$	9.7743$e-11$
0.7	7.6735$e-05$	7.0373$e-08$	6.7821$e-11$
0.8	1.4665$e-05$	6.0573$e-08$	1.4743$e-10$
0.9	8.1616$e-05$	1.5553$e-07$	1.2997$e-10$
1.0	8.5267$e-05$	1.5711$e-07$	1.8751$e-10$

Table 2: Absolute errors for Example 2 at different n.

x	$n = 4$	$n = 8$	$n = 12$
0	0	0	0
0.1	2.2012$e-05$	2.3599$e-10$	5.9771$e-17$
0.2	1.5422$e-04$	2.2695$e-10$	1.0337$e-16$
0.3	2.2010$e-04$	2.8800$e-11$	9.2897$e-17$
0.4	1.7484$e-04$	4.1892$e-10$	9.1015$e-17$
0.5	5.7582$e-05$	5.7543$e-11$	1.2640$e-17$
0.6	5.4134$e-05$	2.8830$e-10$	6.0745$e-17$
0.7	8.7363$e-05$	1.1093$e-10$	1.2000$e-16$
0.8	1.3365$e-05$	3.5180$e-10$	7.3259$e-17$
0.9	1.1663$e-04$	9.5773$e-11$	8.8650$e-17$
1.0	1.4003$e-04$	1.4225$e-10$	3.1275$e-17$

Table 3: Absolute errors for Example 3 at different n.

x	$n = 4$	$n = 8$	$n = 12$
0	0	0	0
0.1	5.1479$e-06$	4.8686$e-11$	8.9783$e-18$
0.2	5.4286$e-05$	5.0102$e-11$	1.4736$e-17$
0.3	8.1338$e-05$	1.4487$e-13$	1.4206$e-17$
0.4	6.5339$e-05$	8.6061$e-11$	1.2362$e-17$
0.5	1.7563$e-05$	9.9855$e-12$	1.4139$e-18$
0.6	3.2690$e-05$	7.0793$e-11$	1.0652$e-17$
0.7	5.3816$e-05$	1.2859$e-11$	1.6736$e-17$
0.8	2.9273$e-05$	6.8704$e-11$	1.2782$e-17$
0.9	2.3223$e-05$	3.3909$e-11$	1.1424$e-17$
1.0	3.0485$e-05$	1.5662$e-11$	2.1410$e-18$

using the perturbed Chebyshev-Galerkin and using Galerkin method [1]. Figure 1 shows the absolute errors at different n and Figure 1(b) presents the perturbed Chebyshev-Galerkin method and exact solutions.

Example 2 (see [1, 34]). Consider the Volterra integrodifferential equation

$$\phi'(x) - \int_0^x \phi(t)\,dt = 1 - 2x\sin(x), \quad 0 \le x \le 1,\ \phi(0) = 0, \tag{13}$$

whose exact solution is $\phi(x) = x\cos(x)$.

The numerical results for the absolute errors are displayed in Table 2 for different values of n; comparison of maximum absolute errors is tabulated in Table 5 while Figure 2 exhibits the perturbed Chebyshev-Galerkin and exact solutions and the maximum absolute errors at different n.

Example 3 (see [1, 34]). Consider the Volterra integrodifferential equation

$$\phi'(x) + \int_0^x t\phi(t)\,dt = -1 + \frac{1}{2}x^2 - x\exp(x), \quad 0 \le x \le 1,\ \phi(0) = 0, \tag{14}$$

whose exact solution is $\phi(x) = 1 - \exp(x)$.

The computational results for the absolute errors are summarized in Table 3 for different values of n and comparison of maximum absolute errors are tabulated in Table 5 while Figure 3 exhibits the perturbed Chebyshev-Galerkin and exact solutions and the maximum absolute error at different n.

Example 4 (see [13, 29]). Consider Fredholm-Volterra integrodifferential equation

$$\phi''(x) + x\phi'(x) - x\phi(x) = e^x - 2\sin(x) + \int_{-1}^{1} \sin(x)\,e^t\phi(t)\,dt, \quad -1 \le x \le 1,\ \phi(0) = 1,\ \phi'(0) = 1. \tag{15}$$

The exact solution is $\phi(x) = e^x$.

Table 4: Absolute errors for Example 4 at different n.

x	$n = 4$	$n = 8$	$n = 12$
0	0	0	0
0.1	2.4216$e-05$	5.5536$e-09$	1.1520$e-12$
0.2	5.6674$e-05$	4.4620$e-08$	9.1630$e-12$
0.3	7.5702$e-05$	1.4986$e-07$	3.0638$e-11$
0.4	1.0165$e-04$	3.5091$e-07$	7.1708$e-11$
0.5	1.8402$e-04$	6.7437$e-07$	1.3786$e-10$
0.6	3.8724$e-04$	1.1437$e-06$	2.3382$e-10$
0.7	7.7496$e-04$	1.7782$e-06$	3.6348$e-10$
0.8	1.3927$e-03$	2.5927$e-06$	5.2990$e-10$
0.9	2.2486$e-03$	3.5978$e-06$	7.3533$e-10$
1.0	3.2922$e-03$	4.8012$e-06$	9.8128$e-10$

For numerical results see Tables 4 and 6 while Figure 4 displays the approximant and the maximum errors as compared with the results from the literature [29].

5. Conclusion

This paper has discussed how the perturbed Chebyshev-Galerkin method can be applied for obtaining solutions of integral and integrodifferential equations. The formulation and implementation of the scheme are illustrated. The proposed method was tested using some problems with results. This paper discussed how the integrodifferential equation with variable and constant coefficients can be solved using the perturbed Chebyshev-Galerkin method. Maple and Matlab had been used to obtain the approximate solution and for plotting the graphs, respectively. Numerical results demonstrate that our method is an accurate and reliable numerical technique for solving mth order integrodifferential and integral equations. Finally, because of the accuracy and simplicity of the elegant method presented in this study, we recommend its application in finding the approximate solution to integrodifferential and integral equations.

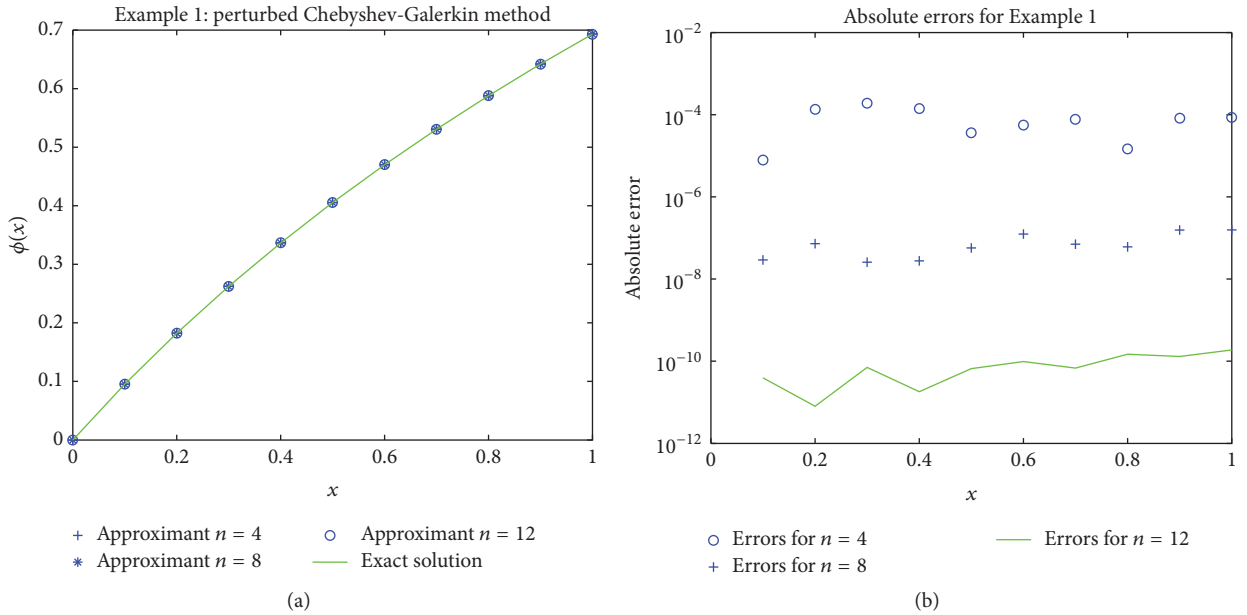


Figure 1: The approximate solution for Example 1 and its absolute errors for different n.

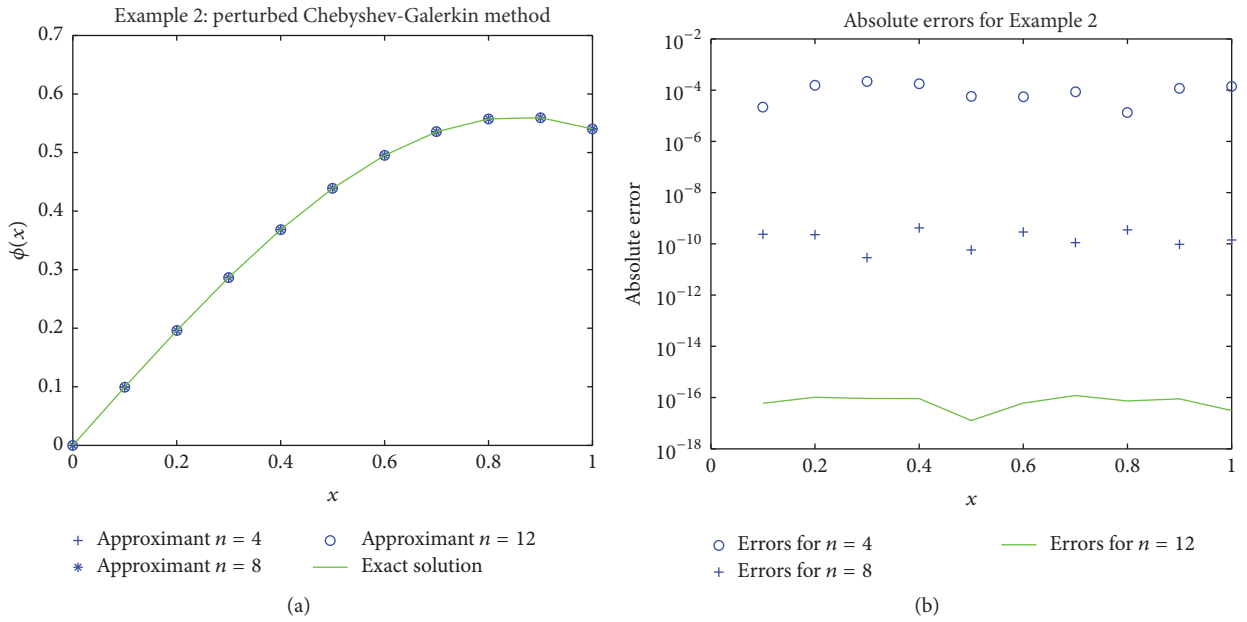


Figure 2: The approximate solution for Example 2 and its absolute errors for different n.

Table 5: Comparison of absolute maximum errors for Examples 1–3 at different n.

n	Example 1		Example 2		Example 3	
	Present method	Galerkin method [1]	Present method	Galerkin method [1]	Present method	Galerkin method [1]
4	$1.9086e-04$	$4.629e-3$	$2.2010e-04$	$2.300e-3$	$8.1338e-05$	$7.204e-4$
8	$1.5711e-07$	$5.611e-6$	$4.1892e-10$	$1.382e-8$	$8.6061e-11$	$2.311e-9$
12	$1.8751e-10$	$5.930e-09$	$1.2000e-16$	$8.087e-15$	$1.6736e-17$	$9.235e-16$

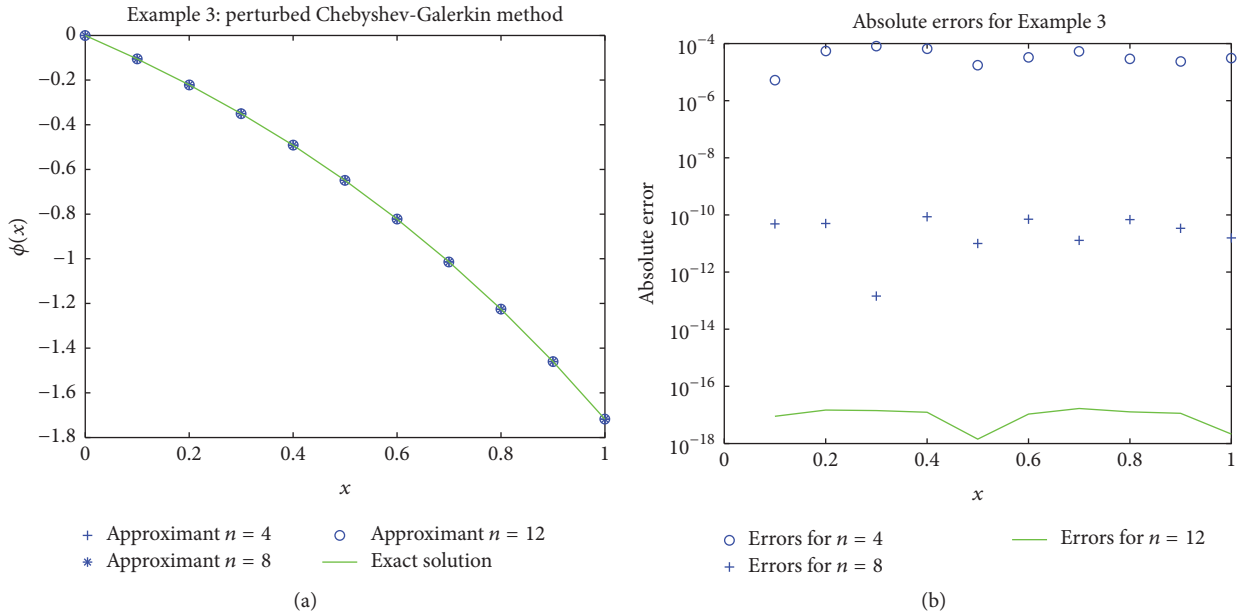


Figure 3: The approximate solution for Example 3 and its absolute errors for different n.

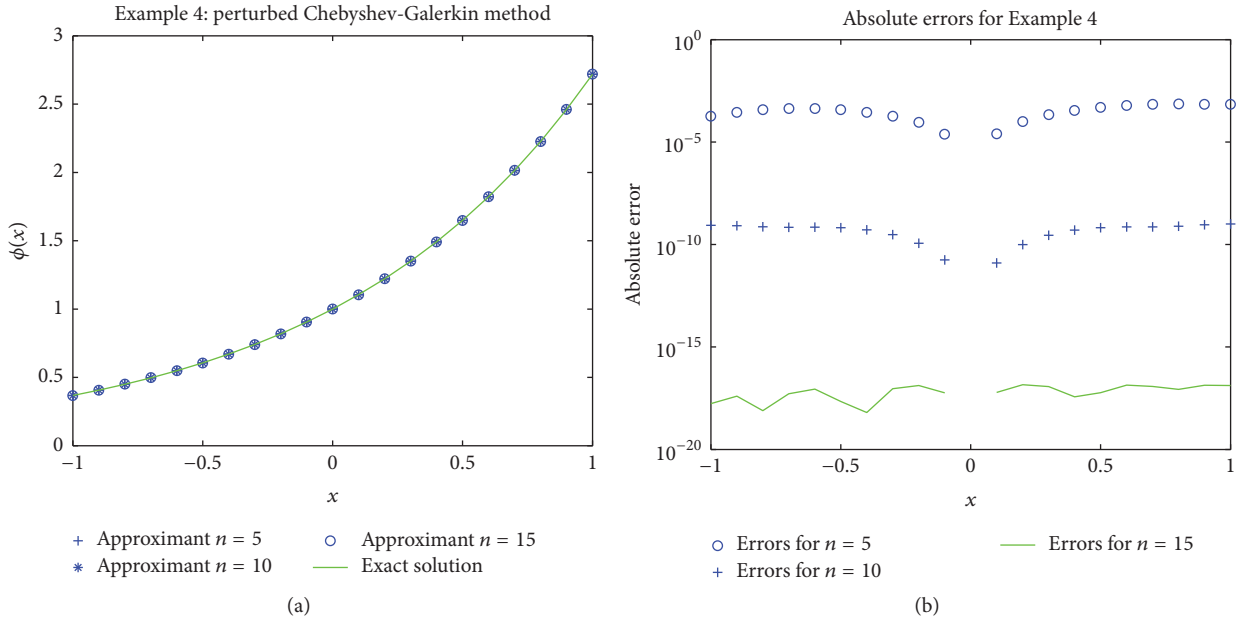


Figure 4: The approximate solution for Example 4 and its absolute errors for different n.

Table 6: Comparison of absolute maximum errors for Example 4 at different n.

n	Present method	Tau method [29]
5	$7.1247e-04$	$3.19e-03$
10	$1.0014e-09$	$2.10e-06$
15	$1.4365e-17$	$7.50e-11$

Competing Interests

The authors declare that they have no competing interests.

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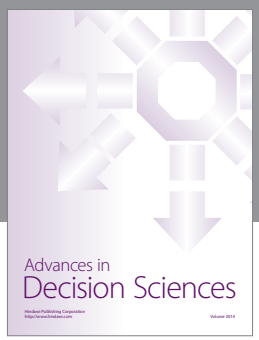
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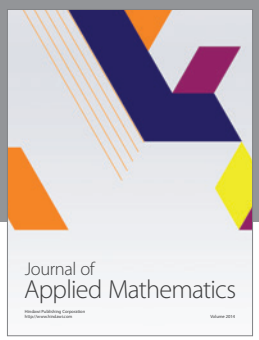
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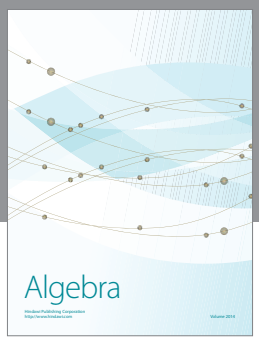
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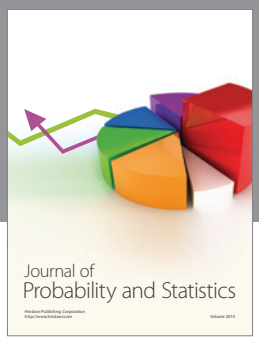
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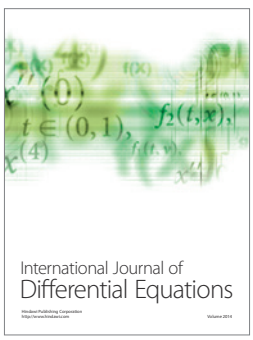


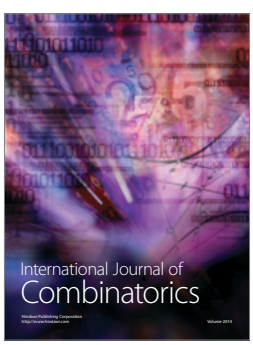


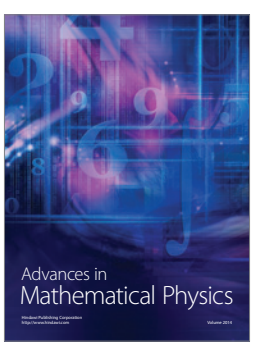


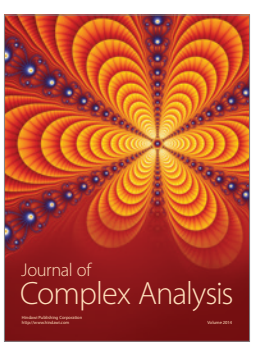


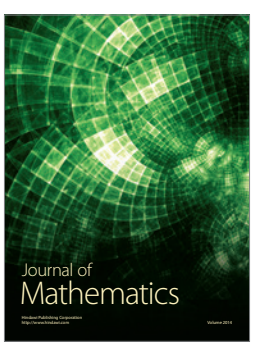


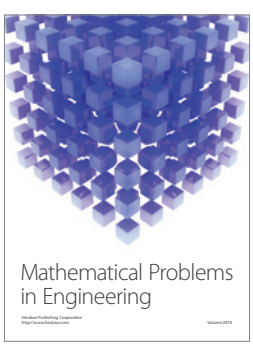


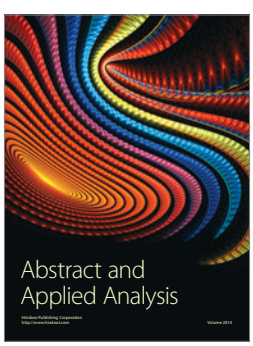


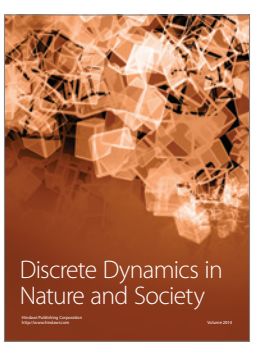


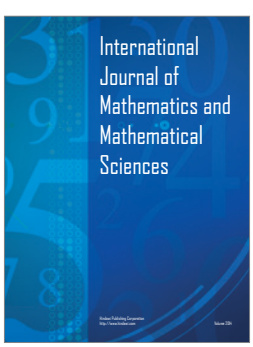






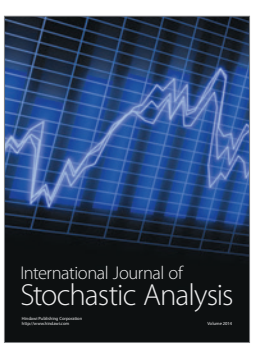














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