Characterization of Optimal Control Model of Sorghum Production

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Abstract
In this work, an optimal control model of sorghum production with all the possible threats at every stages was formulated with the use of biological and botanical controls to maximize sorghum production. The model was characterized through optimal control approach and numerical implication was carried out using fourth order Runge-Kutta method. The results show that for optimum production the two controls must be implemented.

Keywords: Sorghum Production; Threats; Biological Control; Botanical Control, Optimal Control Theory.

Introduction

Sorghum (*Sorghum bicolor L. Moench*) is a very important cereal crop that is widely grown worldwide for food, fiber, forage, ethanol, and sugar production and insect pests are one of the main constraints to small-holder sorghum production in Africa. Sorghum's sweetness attracts not only people, but also pathogens and insects. Africa is one of the main sorghum production areas. Forty two panicle feeding pests have become serious pests of sorghum in West, Central, and South Africa. Thirty nine herbivorous Hemipteran species were found in South Africa, with the most abundant family being *Miridae* and *Lygaeidae*, Kruger *et al.*, (2008). Grasshoppers (*Dichroplus elongatus*) are important sorghum pests in South America. Sorghum shoot fly (*Atherigona soccata*), corn rootworm (*Diabrotica speciosa*), and corn borers (*Ostrinia nubilalis*) are important sorghum pests in Asia, Africa, and Europe. Many of the sorghum pests can damage crops other than sorghum, such as corn, cotton, and millet. Therefore, the host ranges of these sorghum pests need to be carefully considered when rotating crops for managing the insects, Chunshan *et al.*, (2011). In various studies, chemical pesticides have been commonly used to control pest populations, however, the application of these chemical insecticide have many disastrous consequences on the surrounding environment, such as pollution, soil fertility damage and reduction of pest's natural enemies, they also have negative effects on humans and animals, such as various dangerous disease like cancer and disability, Rafikov and Balthazar (2005), Shim *et al.*, (2009). The integrated pest management (IPM) approach in maintaining pest popullation below economic injury level (EIL) is neccesary in order to reduce hazards associated with chemical application in stored cereals, Hill (1997). This leads to a search for eco-friendly methods such as the use of botanicals in the control of insect pest, Suleiman and Rugumamu (2017). Insect Pests have also been controlled biologically, Aderinto *et al.* (2013), Kennis *et al.* (2019), Tiroesele *et al.* (2015), by the introduction of pest natural enemy or predator to reduce numbers or elimination of pest organism by interference with their ecology, though biological control like others can have side effects on biodiversity through attacks on non-targeted species.

Therefore, strategies to prevent and control are required to better prevent pest and the transmission of plant disease at lower cost so also less disastrous. The main interest of this paper is to employ the use of biological and botanical control for optimumality. Many researcher have worked on sorghum analysis, Chuc *et al.* (2013) explore genetic variability for effect of high temperature on crop growth development, in nitro pollen germination and seed set. Leonard and Sivanadane (2016) stated the efficacy, advantages and safety of using biological agents to suppress and control damage done to crops by insects. According to Mofokeng and Shargie (2016) production of sorghum is hindered by abiotic and biotic stress factors, among them birds are a serious pest and they limit grain production from sorghum. With different factors influencing the ability of the birds to damage the crop plant, various methods have been employed to control birds including the use of chemicals, repellent, bird scaring, host pest resistance etc. Their results shows that host plant resistance can be effective method to control due to the presence of tannins in bitter sorghum types.

In another development, a lot of Researchers have worked on optimal control model, Getachew *et al.* (2017) proposed and analyzed a compartmental non-linear deterministic mathematical model for typhoid fever outbreak and optimal control strategies in a community with varying population. Aderinto and Bamigbola (2017) proposed a mathematical expression that represent the dynamics of air pollution associated with electric

power generating industries (atmospheric co2 in particular) using an optimal control theory approach. Anggriani *et al.* (2017) also discussed a mathematical model of plant disease with effect of fungicide. Jing and Suxia (2018) formulate a model for hepatitis B virus with control strategies of new borne vaccination and treatment. A mathematical model that captures the essential dynamics of waterborne disease transmission under the assumption of a homogeneously mixed population was formulated by Obiora and Kevin (2018). However, in view of the surveyed literature, little or no study has been done to capture optimal control of possible threats at every stages of a sorghum plant. Thus in this paper, the interest is to formulate an optimal control model to study the dynamics of possible threats at all stages of growth of sorghum plant using biological and botanical controls in an attempt to maximize sorghum production.

Model Formulation

An optimal control model with two controls is formulated for the seed/plant and insect population.The seed/plant population is classified into Seed Sc(t), Germination G(t), Crop C(t) and Harvest (H), while the insect population is categorized into two: the beneficial insect population, $P_v(t)$, which could there after become harmful to sorghum seeds and pest population $I_v(t)$ right from time, harmful to sorghum. Λ_c is the quantity of replenishing seed for plantation, $\frac{\beta_1 I_v S_v}{N+bI_v}$ is the force of infection of seed by pest. γ and μ are the removal rate from germination class where γ is the proportion that produces crop and μ denote natural bad seeds. εC is the quantity of matured crop that farmers will be able to harvest. u_1 is the biological control and u_2 is the botanical control. The model is stated as:

$$J(u) = Min \int_{t_0}^{t_f} (G(t) + I_v(t) + \frac{1}{2}A_1 u_1^2(t) + \frac{1}{2}A_2 u_2^2(t))dt \quad (1)$$

subject to:

$$\left.\begin{aligned} \frac{dS_c}{dt} &= \Lambda_c - \mu S_c - \frac{(1-u_1)\beta_1 I_v S_c}{N+bI_v} \\ \frac{dG}{dt} &= \frac{\beta_1(1-u_1)I_v S_c}{N+bI_v} - (u_1\gamma + \mu)G \\ \frac{dC}{dt} &= \gamma u_1 G - (u_2\varepsilon + \mu)C \\ \frac{dH}{dt} &= \varepsilon u_2 C - \mu H \\ \frac{dP_v}{dt} &= \Lambda_p - \mu_1 P_v - \alpha P_v \\ \frac{dI_v}{dt} &= \Lambda_v + \alpha P_v - \mu_1 I_v \end{aligned}\right\} \quad (2)$$

where $S_c(0) = S_{c_0} \geq 0$, $G(0) = G_0 \geq 0$, $C(0) = C_0 \geq 0$, $H(0) = H_0 \geq 0$ and $P_v(0) = P_{v0} \geq 0, I_v(0) = I_{v0} \geq 0$ for all t and $N_1 = S_c(t) + G(t) + C(t) + H(t), N_2(t) = P_v(t) + I_v(t)$, $0 \leq u_i(t) \leq 1$, $i = 1,2.$, A_1 and A_2 are weight function associated to U_1 and U_2 respectively.

Table 1: Description of Parameters and Variables. The values are hypothetical

Parameter/ variables	Description	Value
Λ_c	Proportion of seed replenishing	50
β_1	Probability of successful harm on sorghum seed	[0,1]
N	Total number of seed	80
γ	Percentage of germinating seed producing crop	0.6
ε	Fraction of crop which successfully harvest	0.7
μ	Amount of natural bad seed/natural loss on seeds	0.1
Λ_p	Relative recruitment rate for insect with both negative and positive impact	15
Λ_v	Relative recruitment rate for insect with negative impact	50
bI_v	Proportion of pest interacting/having contact with seeds	5.1421
μ_1	Natural death rate for insects	0.1
α	Progressive of destructive insects	0.02
S_c	Sorghum Seeds	50
G	Germination Stage	35
C	Cropping	30
H	Harvest	20

P_v	Beneficial Pest	7.5
I_v	Pest	10.285

Characterization of Optimal Control Problem

Optimal control can be described as the process of finding the extreme value of control and the state trajectories for a dynamical system over a period of time. Many researchers referred to Pontryagin Maximum Principle as a classical result from optimal control theory that provides a condition for an Optimal control problems to have optimal solution(s), (Tragler *et al.* (2008); Yang *et al.*, (2012), Hem *et al.* (2015), Muhammad *et al.* (2016)). The necessary condition can be generated by forming Hamiltonian H, which is stated without prove as follow:

Theorem 1: *Tragler et al. (2008)*

If $u^*(t)$ and $x^*(t)$ are optimal for

$$Max_u \int_{t_0}^{t_f} f(t,x(t),u(t))dt$$

subject to:

$$x'(t) = g(t,x(t),u(t)),$$

$$x(t_0) = x_0 \text{ and } x(t_f) \text{ free,}$$

then there exists a piecewise differentiable adjoint variable $\lambda(t)$ such that

$$H(t,x^*(t),u(t),\lambda(t)) \leq H(t,x^*(t),u^*(t),\lambda(t))$$

$\forall$ control u at each time t, where the Hamiltonian (H) is

$$H = f(t,x(t),u(t)) + \lambda(t)g(t,x(t),u(t)) \quad (3)$$

and

$$\lambda'(t) = -\frac{\partial H(t,x^*(t),u^*(t),\lambda(t))}{\partial x} \quad (4)$$

The control ($u^*(t)$) can be found through differentiation; $\frac{\partial H}{\partial u} = 0$ at $u^*(t)$, which mostly depends on the adjoint variables. Then, the $u^*(t)$ obtained will be substituted in the state variables.

Theorem 2: *Let $S_c^*(t), G^*(t), C^*(t), H^*(t), P_v(t)$ and $I_v^*(t)$ be optimal state solution with associated optimal control variable $u_i^*(t)$, $0 \leq u_i(t) \leq 1$, $i = 1,2$ for the optimal control problem* (1) *and* (2). *Then, there exists adjoint variables λ_i for $i = 1,2,\ldots,6$ that satisfy:*

$$\begin{aligned}
\lambda_{1,}(t) &= (\tfrac{\mu+\beta_1(1-u_1)I_v}{N+bI_v})\lambda_1 - (\tfrac{\beta_1(1-u_1)I_v}{N+bI_v})\lambda_2 \\
\lambda_{2,}(t) &= -1 + (u_1\gamma + \mu)\lambda_2 - (u_1\gamma)\lambda_3 \\
\lambda_{3,}(t) &= (u_2\varepsilon + \mu)\lambda_3 + u_2\varepsilon\lambda_4 \\
\lambda_{4,}(t) &= \mu\lambda_4 \\
\lambda_{5,}(t) &= (\mu_1 + \alpha)\lambda_5 - \alpha\lambda_6 \\
\lambda_{6,}(t) &= -1 + \mu_1\lambda_6 + (\tfrac{\beta_1(1-u_1)S_c}{N+bI_v})\lambda_1 - (\tfrac{\beta_1(1-u_1)S_c}{N+bI_v})\lambda_2
\end{aligned} \quad (5)$$

with transversality conditions

$$\lambda_i(t_f) = 0, \quad \text{for} \quad i = 1,2,\ldots,6 \quad (6)$$

Furthermore, the optimal control $u_i^*(t)$ for $i = 1,2$ is given as

$$u_1^*(t) = max\{min\{\frac{\beta_1 I_v S_c(\lambda_2-\lambda_1)-\gamma G(\lambda_2-\lambda_3)(N+bI_v)}{A_1(N+bI_v)}, 0.9\},0\} \quad (7)$$

and

$$u_2^*(t) = max\{min\{\frac{\varepsilon(\lambda_3-\lambda_4)C}{A_2}, 0.9\},0\} \quad (8)$$

Proof: Let the Hamiltonian, H be defined by:

$$H(t) = L(G,I_v,u_1,u_2) + \lambda_1\frac{dS_c}{dt} + \lambda_2\frac{dG}{dt} + \lambda_3\frac{dC}{dt} + \lambda_4\frac{dH}{dt} + \lambda_5\frac{dP_v}{dt} + \lambda_6\frac{dI_v}{dt} \quad (9)$$

taking negative partial derivative of the Hamiltonian defined in equation (9), each of the state variables return the adjoint expression in the theorem (3.2). Also, the condition for the optimality is $\frac{\partial H}{\partial u_i} = 0$,for $i = 1,2$ and simplification establish the results in (7) and (8) and thus, the solution.

Numerical Application

Most real life problems do not have exact solution or difficult to find, hence, necessitate employment of numerical scheme. In this work, Classical Runge-Kutta algorithm is employed, its classical forward and backward fourth order is used which sometime refers to as sweep method (RK45). The method is known to have greater accuracy and no derivatives are required, Lenhart and Workman (2007) and Garrett (2015). The method was implemented on Maple 18.

Transformation of Optimal Control Problem Equations to Runge-Kutta Scheme

Consider the state equation in (2) and adjoint equation (5), by applying the Runge-Kutta forward and backward method to state and adjoint equation respectively, we have;

$$\left.\begin{aligned}
&\frac{dSc}{dt} = \Lambda_c - \mu S_c - \frac{(1-u_1)\beta_1 I_v S_c}{N+bI_v}\\
&\frac{dG}{dt} = \frac{\beta_1(1-u_1)I_v S_c}{N+bI_v} - (u_1\alpha + \mu)G\\
&\frac{dC}{dt} = \gamma u_1 G - (u_2\varepsilon + \mu)C\\
&\frac{dH}{dt} = u_2\varepsilon C - \mu H\\
&\frac{P_v}{dt} = \Lambda_p - \mu_1 P_v - \alpha P_v\\
&\frac{dI_v}{dt} - \Lambda_v + \alpha P_v + \mu_1 I_v\\
&\frac{d\lambda_1}{dt} = \left(\frac{\mu+\beta_1(1-u_1)I_v}{N+bI_v}\right)\lambda_1 - \left(\frac{\beta_1(1-u_1)I_v}{N+bI_v}\right)\lambda_2\\
&\frac{d\lambda_2}{dt} = -1 + (u_1\alpha + \mu)\lambda_2 - u_1\alpha\lambda_2\\
&\frac{d\lambda_3}{dt} = (u_2\varepsilon + \mu)\lambda_3 + u_2\varepsilon\lambda_4\\
&\frac{d\lambda_4}{dt} = \mu\lambda_4\\
&\frac{d\lambda_5}{dt} = (\mu_1 + \alpha)\lambda_5 - \alpha\lambda_6\\
&\frac{d\lambda_6}{dt} = -1 + \mu_1\lambda_6 + \left(\frac{\beta_1(1-u_1 S_c)}{N+bI_v}\right)(\lambda_1 - \lambda_2)
\end{aligned}\right\} \; OCP\ canonical\ equation \qquad (10)$$

Transforming the state equation using Runge-Kutta forward scheme to obtain:

$$S_{c,i+1} = S_{c,i} + \frac{\Delta t}{6}(K_1^{S_c} + 2K_2^{S_c} + 2K_3^{S_c} + k_4^{S_c}) \qquad (11)$$

Where

$$K_1^{Sc} = f(t_i, S_{c,i}, I_{v,i}) = \Lambda_c - \left(\mu + \frac{[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})]\beta_1 I_{v,i}}{N+bI_{v,i}}\right)S_{c,i}$$

$$K_2^{Sc} = f\left(t_i + \frac{\Delta t}{2}, S_{c,i} + \frac{\Delta t}{2}K_1^{S_c}, I_{v,i} + \frac{\Delta t}{2}K_1^{I_v}\right)$$

$$= \Lambda_c - \left(\mu + \frac{[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})]\beta_1(I_{v,i}+\frac{\Delta t}{2}K_1^{I_v})}{N+b(I_{v,i}+\frac{\Delta t}{2}K_1^{I_v})}\right)(S_{c,i} + \frac{\Delta t}{2}K_1^{S_c})$$

$$K_3^{Sc} = f\left(t_i + \frac{\Delta t}{2}, S_{c,i} + \frac{\Delta t}{2}K_2^{S_c}, I_{v,i} + \frac{\Delta t}{2}K_2^{I_v}\right)$$

$$= \Lambda_c - \left(\mu + \frac{[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})]\beta_1(I_{v,i}+\frac{\Delta t}{2}K_2^{I_v})}{N+b(I_{v,i}+\frac{\Delta t}{2}K_2^{I_v})}\right)(S_{c,i} + \frac{\Delta t}{2}K_2^{S_c})$$

$$K_4^{Sc} = f\left(t_i + \Delta t, S_{c,i} + \Delta t K_3^{S_c}, I_{v,i} + \Delta t K_3^{I_v}\right)$$

$$= \Lambda_c - \left(\mu + \frac{[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})]\beta_1(I_{v,i}+\Delta t K_3^{I_v})}{N+b(I_{v,i}+\Delta t K_3^{I_v})}\right)(S_{c,i} + \Delta t K_3^{S_c})$$

and

$$G_{i+1} = G_i + \frac{\Delta t}{6}(k_1^G + 2K_2^G + 2K_3^G + K_4^G) \quad (12)$$

Where

$$K_1^G = f(t_i, S_{c,i}, I_{v,i})$$
$$= \frac{\beta_1[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})](I_{v,i}S_{c,i})}{N+bI_v} - (u_1\gamma + \mu)G_i$$
$$K_2^G = f\left(t_i + \frac{\Delta t}{2}, S_{c,i} + \frac{\Delta t}{2}K_1^{S_c}, I_{v,i} + \frac{\Delta t}{2}K_1^{I_v}, G_i + \frac{\Delta t}{2}K_1^G\right)$$
$$= \frac{\beta_1[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})](I_{v,i}+\frac{\Delta t}{2}K_1^{I_v})(S_{c,i}+K_1^{S_c}\frac{\Delta t}{2})}{N+b(I_{v,i}+K_1^{I_v}\frac{\Delta t}{2})} - (u_1\gamma + \mu)(G_i + \frac{\Delta t}{2}K_1^G)$$
$$K_3^G = f\left(t_i + \frac{\Delta t}{2}, S_{c,i} + \frac{\Delta t}{2}K_2^{S_c}, I_{v,i} + \frac{\Delta t}{2}K_2^{I_v}, G_i + \frac{\Delta t}{2}K_2^G\right)$$
$$= \frac{\beta_1[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})](I_{v,i}+\frac{\Delta t}{2}K_2^{I_v})(S_{c,i}+\frac{\Delta t}{2}K_2^{S_c})}{N+b(I_{v,i}+\frac{\Delta t}{2}K_2^{I_v})} - (u_1\gamma + \mu)(G_i + \frac{\Delta t}{2}K_2^G)$$
$$K_4^G = f\left(t_i + \Delta t, S_{c,i} + \Delta t K_3^{S_c}, I_{v,i} + \Delta t K_3^{I_v}, G_i + \Delta t K_3^G\right)$$
$$= \beta_1[1 - \frac{1}{2}(u_{1,i} + u_{1,i+1})](I_{v,i} + \Delta t K_3^{I_v})(S_{c,i} + \Delta t K_3^{S_c}) - (u_1\gamma + \mu)(G_i + \Delta t K_3^G)$$

Similarly,

$$C_{i+1} = C_i + \frac{\Delta t}{6}(K_1^c + 2K_2^c + 2K_3^c + K_4^c) \quad (13)$$

Where

$$K_1^C = f(t_i, G_i, C_i) = u_1\gamma G_i - (u_2\varepsilon + \mu)C_i$$
$$K_2^C = f\left(t_i + \frac{\Delta t}{2}, G_i + \frac{\Delta t}{2}K_1^G, C_i + \frac{\Delta t}{2}K_1^C\right) = u_1\gamma(G_i + \frac{\Delta t}{2}K_1^G) - (u_2\varepsilon + \mu)(C_i + \frac{\Delta t}{2}K_1^c)$$
$$K_3^C = f\left(t_i + \frac{\Delta t}{2}, G_i + \frac{\Delta t}{2}K_2^G, C_i + \frac{\Delta t}{2}K_2^C\right) = u_1\gamma(G_i + \frac{\Delta t}{2}K_2^G) - (u_2\varepsilon + \mu)(C_i + \frac{\Delta t}{2}K_2^c)$$
$$K_4^C = f(t_i + \Delta t, G_i + \Delta t K_3^G, C_i + \Delta t K_3^C) = u_1\gamma(G_i + \Delta t K_3^G) - (u_2\varepsilon + \mu)(C_i + \Delta t K_3^c)$$

similarly,

$$H_{i+1} = H_i + \frac{\Delta t}{6}(K_1^H + 2K_2^H + 2K_3^H + K_4^H) \quad (14)$$

Where

$$K_1^H = f(t_i, C_i, H_i) = u_2\varepsilon C_i - \mu H_i$$
$$K_2^H = f\left(t_i + \frac{\Delta t}{2}, C_i + \frac{\Delta t}{2}K_1^C, H_i + \frac{\Delta t}{2}K_1^H\right) = u_2\varepsilon(C_i + \frac{\Delta t}{2}K_1^C) - \mu(H_i + \frac{\Delta t}{2}K_1^H)$$
$$K_3^H = f\left(t_i + \frac{\Delta t}{2}, C_i + \frac{\Delta t}{2}K_2^C, H_i + \frac{\Delta t}{2}K_2^H\right) = u_2\varepsilon(C_i + \frac{\Delta t}{2}K_2^C) - \mu(H_i + \frac{\Delta t}{2}K_2^H)$$
$$K_4^H = f(t_i + \Delta t, C_i + \Delta t K_3^C, H_i + \Delta t K_3^H) = u_2\varepsilon(C_i + \Delta t K_3^C) - \mu(H_i + \Delta t K_3^H)$$

Furthermore,

$$P_{v,i+1} = P_{v,i} + \frac{\Delta t}{6}(K_1^{P_v} + 2K_2^{P_v} + 2K_3^{P_v} + K_4^{P_v}) \quad (15)$$

Where

$$K_1^{P_v} = f(t_i, P_{v,i}) = \wedge_p - (\mu_1 + \alpha)P_{v,i}$$
$$K_2^{P_v} = f\left(t_i + \frac{\Delta t}{2}, P_v + \frac{\Delta t}{2}K_1^{P_v}\right) = \wedge_p - (\mu_1 + \alpha)(P_{v,i} + \frac{\Delta t}{2}K_1^{P_v})$$
$$K_3^{P_v} = f\left(t_i + \frac{\Delta t}{2}, P_v + \frac{\Delta t}{2}K_2^{P_v}\right) = \wedge_p - (\mu_1 + \alpha)(P_{v,i} + \frac{\Delta t}{2}K_2^{P_v})$$
$$K_4^{P_v} = f\left(t_i + \Delta t, P_v + \Delta t K_3^{P_v}\right) = \wedge_p - (\mu_1 + \alpha)(P_{v,i} + \Delta t K_3^{P_v})$$

and, for the last state variable,

$$I_{v,i+1} = I_{v,i} + \frac{\Delta t}{6}(K_1^{I_v} + 2K_2^{I_v} + 2K_3^{I_v} + K_4^{I_v}) \quad (16)$$

Where

$$K_1^{I_v} = f(t_i, I_{v,i}, P_{v,i}) = \wedge_v + \alpha P_{v,i} - \mu_1 I_{v,i}$$
$$K_2^{I_v} = f\left(t_i + \frac{\Delta t}{2}, I_v + \frac{\Delta t}{2}K_1^{I_v}, P_v + \frac{\Delta t}{2}K_1^{P_v}\right) = \wedge_v + \alpha(P_{v,i} + \frac{\Delta t}{2}K_1^{P_v}) - \mu_1(I_{v,i} + \frac{\Delta t}{2}K_1^{I_v})$$
$$K_3^{I_v} = f\left(t_i + \frac{\Delta t}{2}, I_v + \frac{\Delta t}{2}K_2^{I_v}, P_v + \frac{\Delta t}{2}K_2^{P_v}\right) = \wedge_v + \alpha(P_{v,i} + \frac{\Delta t}{2}K_2^{P_v}) - \mu_1(I_{v,i} + \frac{\Delta t}{2}K_2^{I_v})$$
$$K_4^{I_v} = f\left(t_i + \Delta t, I_v + \Delta t K_3^{I_v}, P_v + \Delta t K_3^{P_v}\right) = \wedge_v + \alpha(P_{v,i} + \Delta t K_3^{P_v}) - \mu_1(I_{v,i} + \Delta t K_3^{I_v})$$

Transforming the adjoint equation using the backward Runge-Kutta method to obtain:

$$j = N + 2 - i$$

$$K_1 = f(t_j, \lambda_{1,j}, \lambda_{2,j}) = \left(\frac{\mu+[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})]\beta_1 I_{v,j}}{N+bI_{v,j}}\right)\lambda_{1,j} - \left(\frac{\beta_1[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})]I_{v,j}}{N+bI_{v,j}}\right)\lambda_{2,j}$$

$$K_2 = f\left(t_j - \frac{h}{2}, \lambda_{1,j} - \frac{h}{2}K_1, \lambda_{2,j} - \frac{h}{2}K_1\right)\left(\frac{\mu+\beta_1(1-\frac{1}{2}(u_{1,i}+u_{1,i+1})(I_{v,j}+I_{v,j-1})}{N+b(I_{v,j}+I_{v,j-1})}\right) * 0.5 * \left[\left(\lambda_{1,j} - \frac{h}{2}K_1\right) - \left(\lambda_{2,j} - \frac{h}{2}K_1\right)\right]$$

$$K_3 = f\left(t_j - \frac{h}{2}, \lambda_{1,j} - \frac{h}{2}K_2, \lambda_{2,j} - \frac{h}{2}K_2\right)\left(\frac{\mu+\beta_1[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})]*0.5*(I_{v,j}+I_{v,j-1})}{N+b(I_{v,j}+I_{v,j-1})}\right)\left[\left(\lambda_{1,j} - \frac{h}{2}K_2\right) - \left(\lambda_{2,j} - \frac{h}{2}K_2\right)\right]$$

$$K_4 = f(t_j - h, \lambda_{1,j} - hK_3, \lambda_{2,j} - hK_3)\left(\frac{\mu+\beta_1[1-\frac{1}{2}(u_{1,i}+u_{1,i+1})]*0.5*(I_{v,j-1})}{N+bI_{v,j-1}}\right)\left[\left(\lambda_{1,j} - hK_3\right) - \left(\lambda_{2,j} - hK_3\right)\right]$$

$$\lambda_{1,J-1} = \lambda_{1,j} - \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (17)$$

Simulation

Maple 18 was used to solve the transformed optimal control problem and the results are represented by the following graphs (Figures 1 -5).

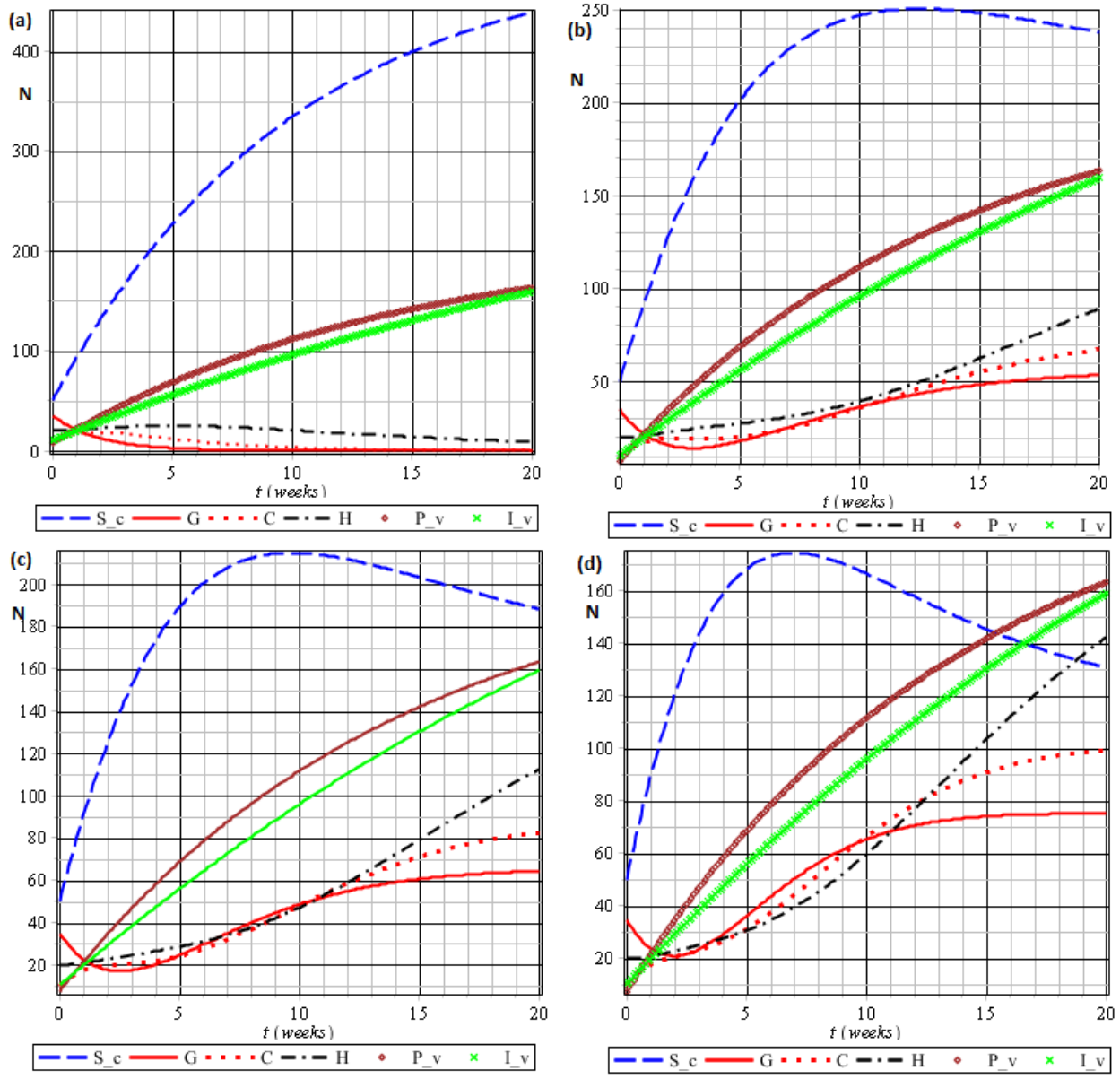
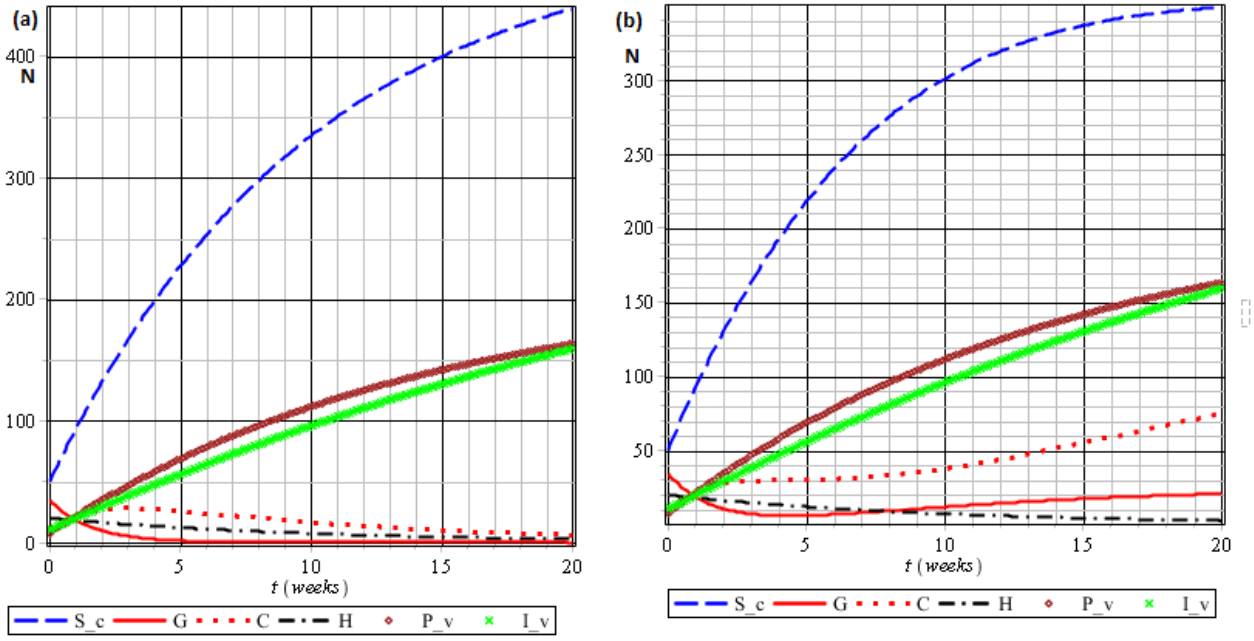


Figure 1: Control implementation for which biological control at 70% and botanical control for 30%



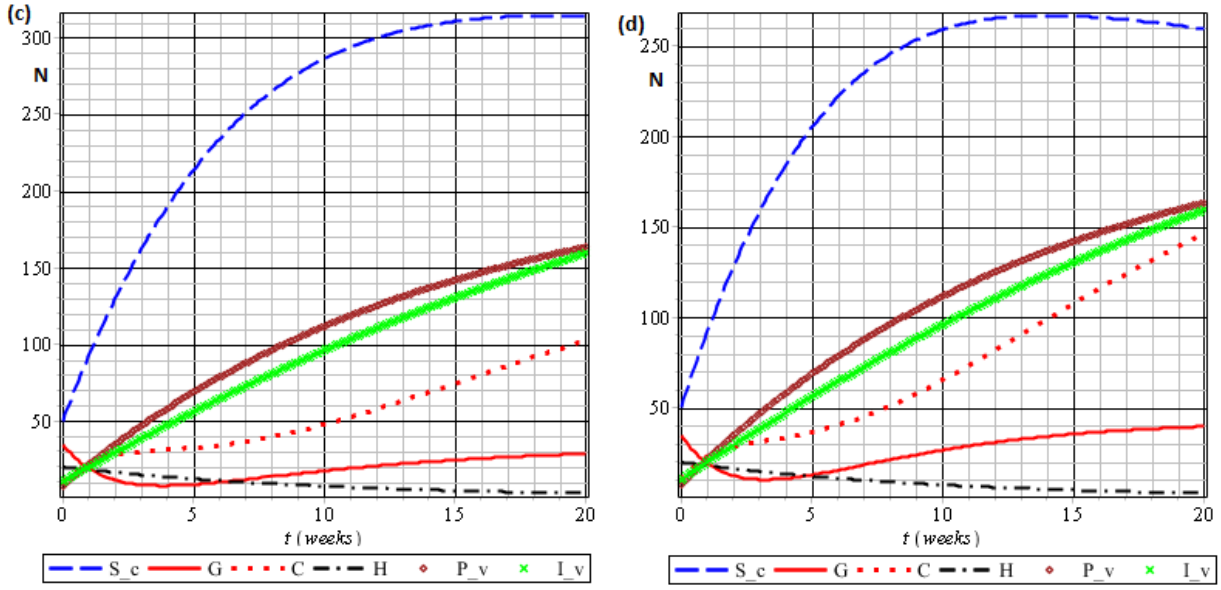


Figure 2: Control implementation for biological control at 90% and botanical control at 0%

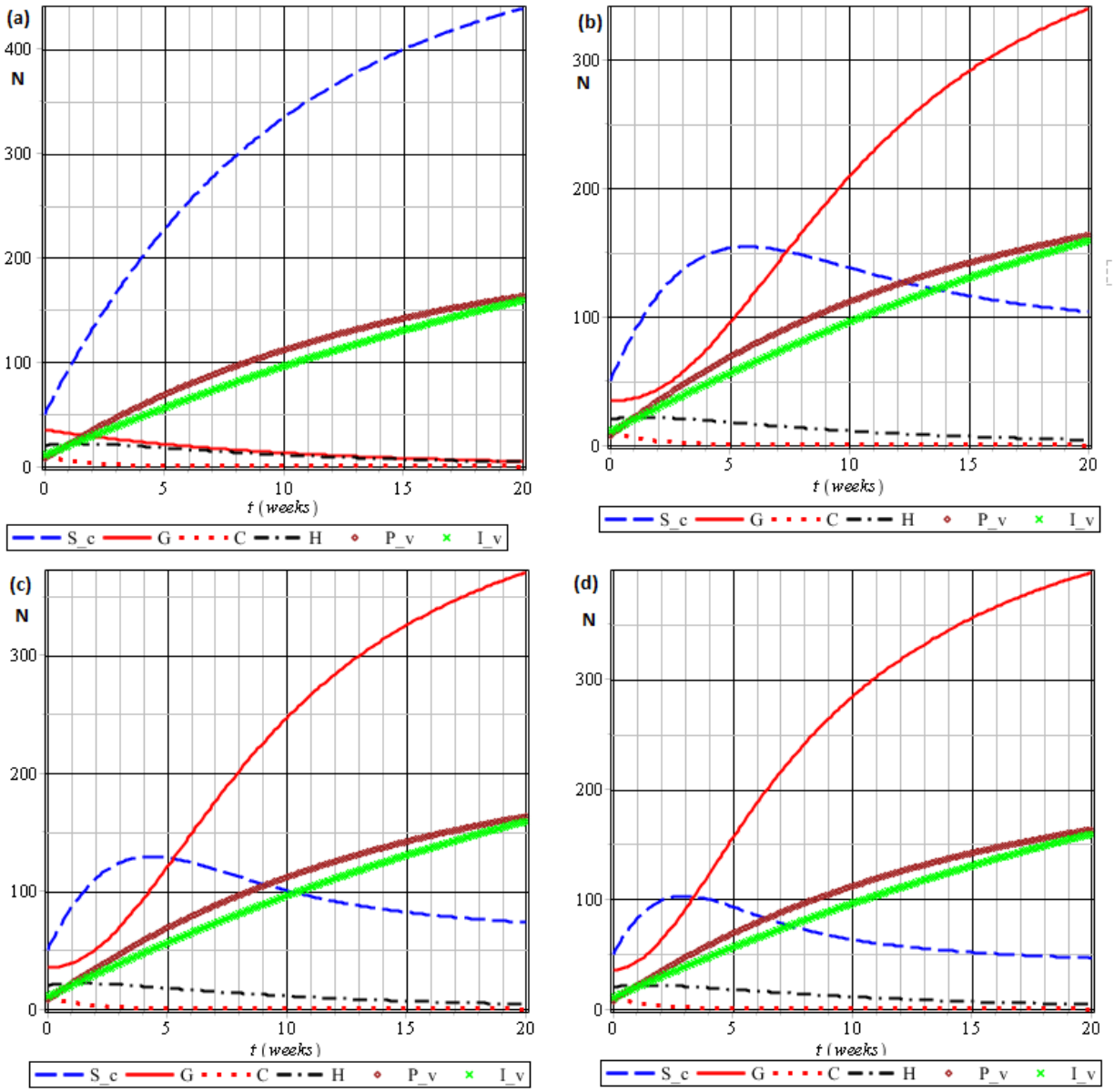


Figure 3: Control implementation for biological control at 0% and botanical control at 90%

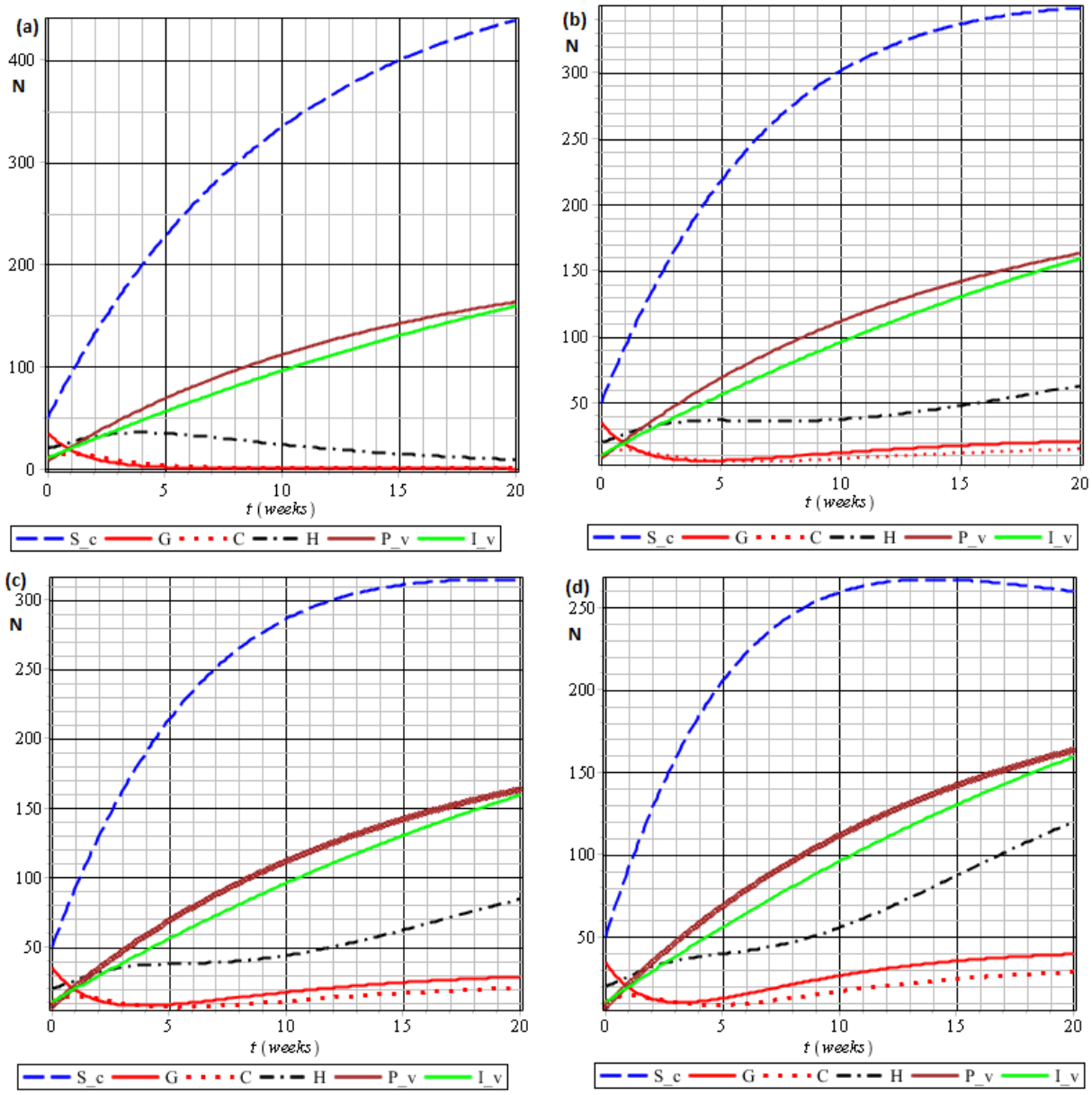


Figure 4: Control implementation for which biological control at 90% and botanical control for 90%

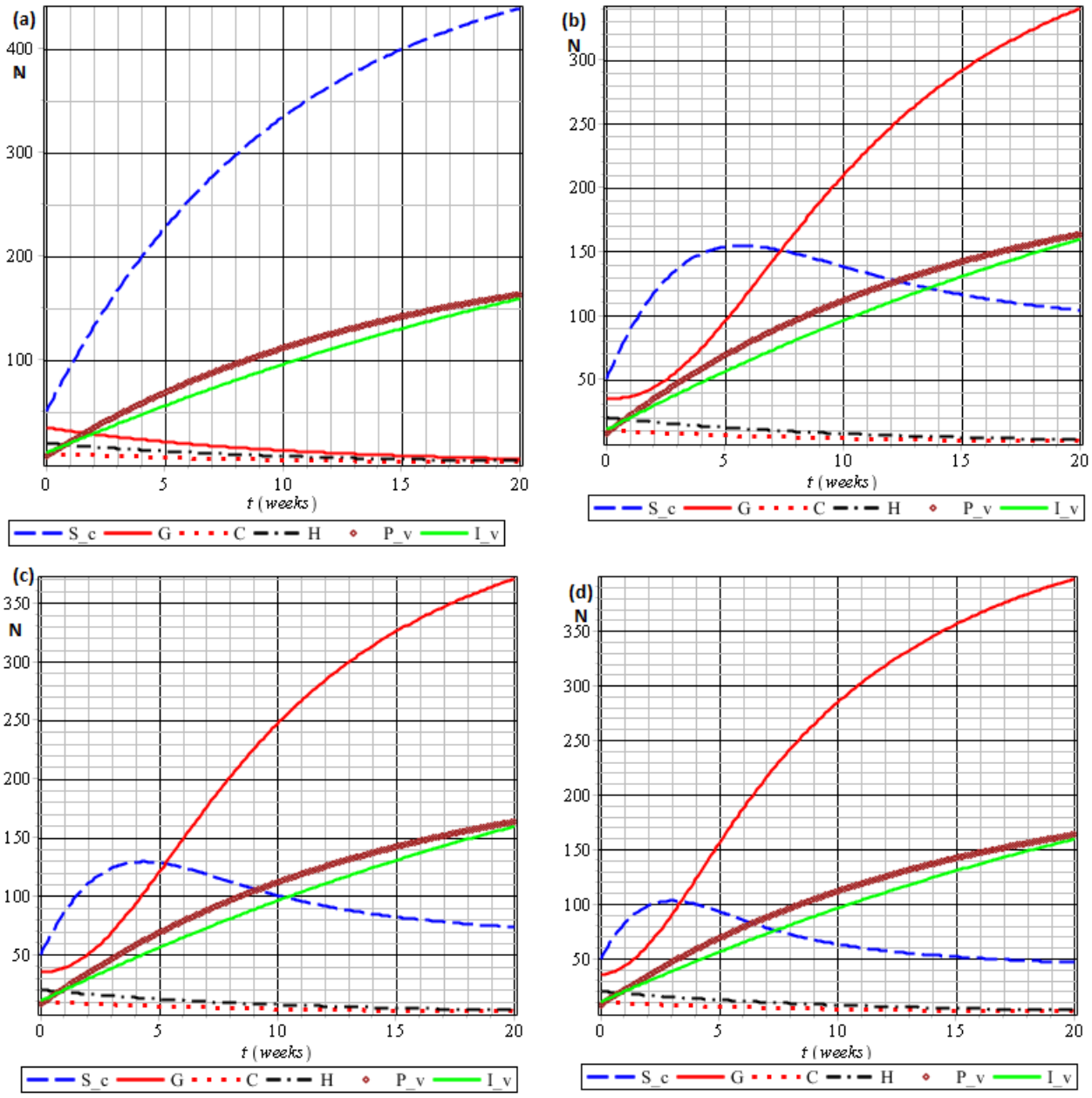


Figure 5: For zero control, biological control at 0% and botanical control at 0%

Discussion of Results

The results obtained show that varying u_1 and u_2($0 \leq u_i(t) \leq 1$, $i = 1,2$) the rate of attack gives likely production. Figure 2 shows the impact of using only biological control, and is not sufficient to obtain optimum production even though there was a little rise in cropping, the harvest was so bad. This shows that biological control alone can have its own side effect. So also in Figure 3, botanical control was implemented at full while there was no biological control at all, it was observed that despite the fact that there was good germination, the cropping and the harvest were at very low rate. It was established that neither biological control nor botanical control is adequate to control sorghum pest. Therefore, there is need to take into consideration both controls

even though the effect of one is needed than the effect of the other. The impact of both controls at full implementation (i.e. biological control at 90% and botanical control at 90%) as seen in 4, is similar to the result when there was no control implementation at all, (i.e. Figure 5, biological control at 0% and botanical control at 0%), this is because even if the controls are not well managed, it will only be as if there was no control implementation. This implies that no controls or excessive implementation of the selected controls can as well have there side effect on the sorghum plant. This call for caution when controls are been implemented. However, there is optimum production if the controls are effectively managed (Figure 1), implementing biological control at 70% and botanical control at 30% returned a better germination, a better cropping and a much better harvest, significantly, there is need for us to take both the control into consideration since considering only one will only yield average production. Hence, so far in the simulations, the scenario for which biological control and botanical control got 70% and 30% implementation respectively gives the maximum production.

Conclusions

A optimal control model to study the dynamics of possible threats at all stages of growth of sorghum plant was formulated with the biological and botanical controls as the optimal controls to minimize pest below injury level in an attempt to maximize sorghum production. Runge-Kutta fourth order method was used to validate the numerical solution. And the results obtained shows that ability to effectively know and manage the controls is key if the farmers must enjoy optimum production and most specific, biological control and botanical control must not exceed 70% and 30% respectively for optimality.

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