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On Some New Results Of A Subclass Of Univalent Functions

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ABSTRACT

In this paper, a generalized class of Spiral-like functions is investigated. Characterization properties for the class is established. Results obtained generalize some well known results.

Keywords: Spiral-like functions, Univalent functions, Salagean Operator.

1. INTRODUCTION

Let A denotes the class of functions $f(z)$ that are analytic in the unit disk $U = \{z \in C : |z| < 1\}$. Furthermore, let S be the class of functions in A that are univalent and have the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

For β that is real, $|\beta| < \frac{\pi}{2}$, a function $f(z)$ of the form (1) is said to be spiral-like of order β if and only if

$$\mathrm{Re}\left\{e^{i\beta}\frac{zf'(z)}{f(z)}\right\} > 0 \qquad , \quad z \in U$$

The class of spiral-like functions of order β is denoted as $S_p(\alpha)$ [1].

In 1932, Spacek.L [2] showed that spiral-like functions are univalent functions.

Let $S^*(\alpha)$ and $C(\alpha)$, $0 \le \alpha < 1$ be the classes of starlike functions of order α and convex functions of order α respectively.

i.e

$$S^*(\alpha) = \left\{ f \in S : \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in U \right\}$$

and

$$C(\alpha) = \left\{ f \in S : \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, z \in U \right\}$$

For $\beta = 0, S_p(0) = S^*$ which is the class of starlike functions . For functions $f(z) \in A$, the Salagean differential operator [3] is defined as $D^n : A \to A$

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = zf^1(z)$$
$$D^n f(z) = D[D^{n-1} f(z)], n = 0, 1, 2, 3, \ldots$$

For the function of the form (1),

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad , \quad z \in U$$

Let

$$S_n(\beta, \alpha) = \left\{ f \in S : \mathrm{Re}\left\{ e^{i\beta} \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \alpha \ , \ z \in U \right\}$$

where D^n is the Salagean differential operator with $n \in \aleph \cup 0$, $|\beta| < \frac{\pi}{2}$.

Remarks

(i) *when* $n = 0$, $S_n(\beta, \alpha) = S_p(\alpha)$
(ii) *when* $n = 0$, $\beta = 0$, $S_n(\beta, \alpha) = S^*(\alpha)$
(iii) *when* $n = 1$, $\beta = 0$, $S_n(\beta, \alpha) = C(\alpha)$

2. MAIN RESULTS

THEOREM 2.1

Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$. *If* $\left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| < \alpha \cos\beta$ then $S^*(\alpha)$ then

$f(z) \in S_n(\beta, \alpha)$ *for* $\beta > \cos^{-1}\left(\frac{\alpha}{1-\alpha} \right)$, $|\beta| < \frac{\pi}{2}$.

PROOF:

Suppose

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad and \quad \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| < \alpha \cos\beta$$

then

$$\frac{D^{n+1}f(z)}{D^n f(z)} - 1 = (\alpha\cos\beta)\omega(z), \qquad |\omega| < 1$$

$$\frac{D^{n+1}f(z)}{D^n f(z)} = 1 + (\alpha\cos\beta)\omega(z)$$

$$\mathrm{Re}\frac{e^{i\beta}D^{n+1}f(z)}{D^n f(z)} = \mathrm{Re}\left[e^{i\beta} + (\alpha\cos\beta)e^{i\beta}\omega(z)\right]$$

$$= \cos\beta + \mathrm{Re}(\alpha\cos\beta)e^{i\beta}\omega(z)$$

since $\mathrm{Re}\, z \geq -|z|$,then

$$\mathrm{Re}\frac{e^{i\beta}D^{n+1}f(z)}{D^n f(z)} \geq \cos\beta - \alpha\cos\beta\left|e^{i\beta}\right|\left|\omega(z)\right|$$

$$\geq \cos\beta - \alpha\cos\beta$$

this implies that

$$\mathrm{Re}\frac{e^{i\beta}D^{n+1}f(z)}{D^n f(z)} > \alpha$$

i.e

$$(1-\alpha)\cos\beta > \alpha$$

then

$$\beta > \cos^{-1}\left(\frac{\alpha}{1-\alpha}\right)$$

THEOREM 2.2

$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ *is in* $S_n(\beta,\alpha)$, $0 \leq \alpha < 1$ *if and only if* $\sum_{k=2}^{\infty} k^n(k+2-\alpha)|a_k| < 1-\alpha.$

PROOF

Let $f(z) \in S_n(\beta,\alpha)$ this implies that

$$\operatorname{Re}\frac{e^{i\beta}D^{n+1}f(z)}{D^{n}f(z)}>\alpha$$

$$\left|e^{i\beta}\frac{D^{n+1}f(z)}{D^{n}f(z)}-1\right|<1-\alpha$$

$$\left|\frac{D^{n+1}f(z)}{D^{n}f(z)}\right|<2-\alpha$$

and this gives

$$\frac{\left|z+\sum_{k=2}^{\infty}k^{n+1}a_k z^k\right|}{\left|z+\sum_{k=2}^{\infty}k^{n}a_k z^k\right|}<2-\alpha$$

that is

$$\left|\frac{D^{n+1}f(z)}{D^{n}f(z)}\right|<2-\alpha$$

If

$$1+\sum_{k=2}^{\infty}k^{n+1}|a_k|<\left(1-\sum_{k-2}^{\infty}k^{n}|a_k|\right)(2-\alpha)$$

$$1+\sum_{k=2}^{\infty}k^{n+1}|a_k|+(2-\alpha)\sum_{k=2}^{\infty}k^{n}<2-\alpha$$

this implies that

$$\sum_{k=2}^{\infty}k^{n}(k+2-\alpha)|a_k|<1-\alpha$$

On the other hand suppose that

$$\sum_{k=2}^{\infty} k^n (k+2-\alpha)|a_k| < 1-\alpha$$

$$\left|\frac{e^{i\beta} D^{n+1} f(z)}{D^n f(z)} - 1\right| \le \left|\frac{D^{n+1} f(z)}{D^n f(z)}\right| + 1$$

$$\le \frac{2 + \sum_{k=2}^{\infty} k^n (k-1)|a_k|}{1 - \sum_{k=2}^{\infty} k^n |a_k|}$$

$$\le \sum_{k=2}^{\infty} k^n (k+1)|a_k|$$

$$< \sum_{k=2}^{\infty} k^n (k+2-\alpha)|a_k| < 1-\alpha \qquad , \qquad 0 \le \alpha < 1$$

i.e

$$\left|\frac{e^{i\beta} D^{n+1} f(z)}{D^n f(z)} - 1\right| < 1-\alpha$$

therefore,

$$\mathrm{Re}\,\frac{e^{i\beta} D^{n+1} f(z)}{D^n f(z)} > \alpha$$

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