A Direct Optimal Control of the Jebba Hydropower Station

Olalekan Ogunbiyi
Electrical and Computer Engineering
Kwara State University, Malete
Malete, Nigeria.
biyikan@gmail.com

Cornelius T. Thomas
Electrical and Information Engineering
Achievers University
Owo, Nigeria
corneliustthomas@gmail.com

Benjamin J. Olufeagba
Electrical and Electronics Engineering
University of Ilorin
Ilorin, Nigeria
oliver.habakuk@gmail.com

Ibrahim S. Madugu
Electrical Engineering
Kano University of Science and Technology
Wudil, Nigeria
maduguemir@gmail.com

Busayo H. Adebiyi
Computer Engineering
Ahmadu Bello University
Zaria, Nigeria
badebiyi@abu.edu.ng

Lambe M. Adesina
Electrical and Computer Engineering
Kwara State University, Malete
Malete, Nigeria
lambe.adesina@kwasu.edu.ng

***Abstract* — The optimal power generation from the Jebba hydroelectric power station is subject to the reservoir operating head, weather-related factors, units' availability and system dynamics. In this paper, a computer control system is designed to ensure safe operation and maximize power generation. The controller is an optimal controller, which determines the amount of inflow required to regulate the reservoir operating head. The control law is an optimal control procedure developed around the steepest descent and conjugate gradient algorithm. The algorithms determine the control signal and state trajectories for minimization of a performance index defined for the regulation of the reservoir operating head. The results show that the two techniques are feasible in estimating an optimal inflow needed to move the reservoir operating head from any level to the nominal head. The two techniques were compared under different operating conditions of the hydropower system, the conjugate gradient algorithm performs better in terms of computational time. The control algorithm is recommended for use in the realization of a computer control system for the station.**

Keywords *— Conjugate gradient, differential equation, operating head, optimal control, performance index.*

I. INTRODUCTION

The hydroelectric power system consists of various dynamical responses such that research in this field seems unending. These include the reservoir dynamics, turbine dynamics, generator dynamics, cooling system[1]. The Jebba Hydroelectric Power Station (JHEPS) in Nigeria is not an exception. It is located at relatively on $09^0 08' 08'' N$, $04^0 47' 16'' E$, 103 km downstream of the Kainji Hydroelectric Power Station (KHEPS) [2], [3].

The JHEPS was installed with six units of fixed blade propeller type turbo-alternator and each is rated at 96.4 MW. The nominal operating head is specified at 27.6 m with a 380 m^3/s maximum flow per unit. The turbo-alternators are connected to a transformer, each rated 119 MVA to step the voltage up to 330 kV [3], [4]. While the KHEPS is designed such that the operating head can varies within 24m to 24m, JHEPS was designed in such a manner that the operating head remains relatively constant for system stability and station safety. The nominal operating head is 27.1m while the min net head is 24m and the maximum of 29.3m.



Fig. 1. Plan View of the KHEPS and JHEPS

Electric power generation at JHEPS is highly affected by the seasonal inflow on the River Niger with its white and black flood characteristics, the operation on the upstream reservoir at Kainji, the number of available units at a time and environmental factors that are weather related[5]. In order to ensure an optimal power generation in the presence of varying and uncertain inflow, an exponentially smoothed time series model of the inflow was suggested by Ale et al.[6]. The model could forecast the probable inflow on the River Niger over the year but it cannot handle the variation in reservoir head due to the turbo-alternators (TAs) availability. The failure of a unit at KHEPS reduces the inflow into JHEPS except on free fun, this can lead to a reduction in the number of operating TAs at JHEPS, thus reducing the power generated [3].

In Nwobi-Okoye and Igboanugo [7], a time series analysis of the inflow into KHEPS and the generated power output for a 10 years period resulted in a transfer function. This provided a means of estimating the efficiency of the power generation. Unfortunately, the model formulated using this method is non-causal, hence cannot be used in control design.

To obtain optimal power generation, a statistical model of inflow observation and the use of dynamic programming for optimal policy formulation have been widely presented[8], [9]. Similarly, an artificial neural network model was also presented in [10]. A more suitable technique is needed to handle a nonlinear dynamical system.

The determination of the control input that is needed to force a dynamical system to follow a predefined trajectory and at the same time minimising a defined performance index within the system constraints is one of the problems solved in modern control engineering. Hence, an optimal control problem can be formulated such as to optimise the power generation from JHEPS and ensure a relatively constant operating head. The optimal control, in this case, is the amount of release from the upstream reservoir that will restore the head to the nominal value, whenever there is a deviation. Hence the station can be better managed and generation potential maximised.

The system dynamical model can be linear or nonlinear, there are usually, constraints on the boundary value and sometimes there can be constraints on the system state. The performance index can also be very complicated such that the solution requires sophisticated techniques. The solutions to an optimal control problem are grouped into two: indirect and direct methods[11].

The indirect methods solve the optimal control problem using the Pontryagin maximum principle. It is based on the calculus of variations to construct the formulation of the Hamiltonian function. The Euler-Lagrange equation is then used to obtain the optimal control canonical equations. Solutions to these equations ensure the determination of the optimal control and state trajectories. Usually, these leads to solving a multi-point boundary value problem by advanced numerical techniques involving the computation of partial derivatives. Also, powerful techniques may also be required where there are path inequalities.

In the case of the direct methods, the optimal solution is determined by converting the infinite-dimensional optimization problem to a finite-dimensional optimization problem[12], [13]. The optimal control problems are formulated as a nonlinear programming problem by discretization of the state and the control variables. The solution is then obtained using an appropriate optimization algorithm[14].

In most cases, the solution is numerical such that the algorithm iterates until a certain condition or stopping criteria are met[15]. According to Rao (2009), the numerical methods of solving nonlinear programming problem is categorised into the gradient-based methods and the heuristic-based methods [13], [16].

The steepest descent and conjugate gradient methods are efficient gradient-based optimization algorithm for solving nonlinear programming problem. In this paper, the two methods used as a direct method of solving the optimal control of the JHEPS operating head.

II. The Optimal Control Problem at JHEPS

A typical close loop control system is as shown in Fig. 2, consisting of the desired operating head, the controller, the actuator and the system.

An essential element of solving optimal control is the dynamic equation describing the system JHEPS [17]. The structural model of the JHEPS system is presented in Fig. 3, showing the reservoir, the turbine and the TAs. The electrical power (P) generated is related to the operating head by Equation (1), where n is the number of operating units, η represents the conversion efficiency of the turbo-alternators, ρ stands for the density of water in kg/m^3, A is the effective correctional area at the scroll casing and g is the acceleration due to gravity (m/s^2). Let $x(t)$ represents the operating head h and the system state while $u(t)$ is the net inflow. n is the number of operating units. A_1 is the effective surface area of the reservoir while A_2 represents the effective area of the scroll casing. g stands for the acceleration due to gravity, Q_J represents the inflow into JHEPS, Q_L represents the evaporation loss on JHEPS and Q_s is the spillway discharge from JHEPS.

Equation (1) implied that the power generation depends greatly on the operating head dynamics, hence the dynamical model for the operating head of JHEPS is represented by Equation (2), where α and μ are constants relating to the reservoir[5].

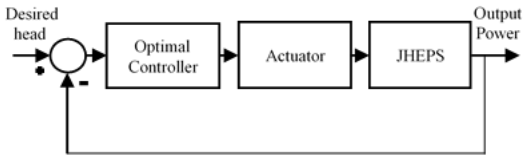


Fig. 2." The structural model of the JHEPS system

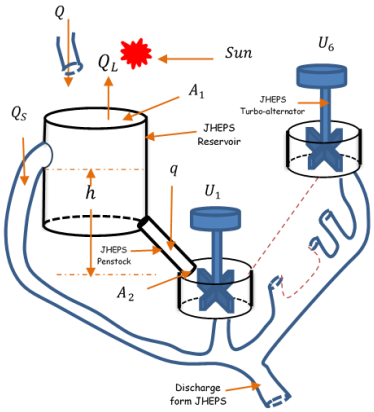


Fig. 3." The structural model of the JHEPS system

$$P = \sqrt{2}\, n\eta\rho A\, g^{3/2}\, x^{3/2}(t) \qquad (1)$$

$$\frac{dx(t)}{dt} = -n\alpha\, x^{1/2}(t) + \mu u(t) \qquad (2)$$

Equation (2) can be written in the standard form for the nonlinear system as shown in equation (3), where f represents a nonlinear function.

$$\dot{x}(t) = \boldsymbol{f}(x(t), u(t), t) \;;\; t_0 \le t \le t_f \qquad (3)$$

There are different kinds of optimal control problem associated with equation (2). The focus here is on the estimation of the control vector $u(t)$ that forces the state trajectories to move from a point $x(t_0)$ to another point $h(t_f)$ within a time frame $(t_0 \rightarrow t_f)$ and at the same time minimizing the performance index $I(x,u,t)$. $I(x,u,t)$ is defined in equation (4), where K_x is a positive scalar constant.

$$I(x,u,t) = min \int_{t_0}^{t_f} \left\{ K_x \big(x(t) - x(T)\big)^2 \right\} dt \qquad (4)$$

The performance index is subject to the following constraints:

$$\dot{x}(t) = \boldsymbol{f}(x(t), u(t), t); \quad t_0 \le t \le t_f$$

$$x(t_0) = x_0 \tag{5}$$

$$x(t_f) = x(T) \tag{6}$$

$$[u_{min}(t) \le u(t) \le u_{max}(t)]$$

Since a direct method of solution is to be used for solving the optimal control problem, the time interval $t_0 \to t_f$ is divided into $N > 1$ intervals, resulting in equation (7). Within these sub-intervals, the infinite-dimensional control signal is approximated to a finite control vector of the form in equation (8). Similarly, the performance index is partitioned to create a vector of the form in equation (9).

$$\pi[t_0, t_f] = t_0 < t_1 < t_2 < \cdots < t_N = t_f \tag{7}$$

$$u(t) = [u_1, u_2, u_3, \dots, u_N] \tag{8}$$

$$I(x, u, t) = [I_1, I_2, I_3, \dots, I_N] \tag{9}$$

Hence the problem has been converted to a nonlinear optimization which is solved using the steepest descent and conjugate gradient algorithms.

III. Optimal Control using a Steepest Descent Algorithm

The steepest descent algorithm is a gradient-based technique used to solve a problem involving the determination of the minimum of a function that is differentiable. It is important that the performance index $I(x, u, t)$ must be differentiable such that the descent direction equals the path opposite $\nabla I(x, u, t)$.

Given an initial control input $u^{k=0}(t)$, the algorithm modifies the control as the iteration k increases until the minimum point is reached where $u^k(t) = u^*(t)$. If the initial guess is far from the feasible region, the algorithm may diverge [16], [18], [19].

At the minimum point, it is expected that;

$$\nabla I^* = \frac{\partial I^*(x,u,t)}{\partial u^*(t)} = \left\| \frac{\partial I_1^*}{\partial u_1^*}, \frac{\partial I_2^*}{\partial u_2^*}, \frac{\partial I_3^*}{\partial u_3^*}, \dots, \frac{\partial I_N^*}{\partial u_N^*} \right\| \approx 0 \tag{10}$$

Let $\boldsymbol{\tau}(u^k)$ be a unit vector along the increasing gradient, the search direction in the opposite. The unit vector is defined as,

$$\boldsymbol{\tau}(u^k) = \frac{\left[\frac{\partial I^k}{\partial u^k}\right]^T}{\left\|\frac{\partial I^k}{\partial u^k}\right\|} = \frac{\left[\frac{\partial I_1^k}{\partial u_1^k}, \frac{\partial I_2^k}{\partial u_2^k}, \frac{\partial I_3^k}{\partial u_3^k}, \dots, \frac{\partial I_N^k}{\partial u_N^k}\right]^T}{\left\|\frac{\partial I_1^k}{\partial u_1^k}, \frac{\partial I_2^k}{\partial u_2^k}, \frac{\partial I_3^k}{\partial u_3^k}, \dots, \frac{\partial I_N^k}{\partial u_N^k}\right\|} \tag{11}$$

Then the next control vector can be computed by equation (12), where ψ^k is step size along the search direction. A line search is usually performed to determine the optimum value for ψ.

$$\begin{bmatrix} u_1^{k+1} \\ u_2^{k+1} \\ u_3^{k+1} \\ \vdots \\ u_N^{k+1} \end{bmatrix} = \begin{bmatrix} u_1^k \\ u_2^k \\ u_3^k \\ \vdots \\ u_N^k \end{bmatrix} - \psi^k \boldsymbol{\tau}(u^k) \tag{12}$$

IV. Optimal Control using a Conjugate Gradient Algorithm

Another gradient-based algorithm for the minimization of a differentiable function is the conjugate gradient algorithm. Lasdon (1967) used this algorithm to solve an unconstrained optimal control problem applied it before it was employed in solving a constrained optimal control problem where a penalty function imposed on the state and control [11], [20]–[22].

The algorithm incorporates the fast characteristics of steepest descent method whereby it determines a feasible solution after a few iterations but uses additional modification in ensuring that it converges to an optimum solution with few iterations.

The steps start by guessing and initial control vector $u^k(t) = [u_1^k, u_2^k, u_3^k, \dots, u_N^k]$, and determine the performance index $I(u_1^k, u_2^k, u_3^k, \dots, u_N^k)$ with the first iteration where $k = 0$.

The next step is to determine the gradient vector g^k.

$$g^k = \nabla I(u^k) = [g_1^k, g_2^k, g_3^k, \dots, g_N^k]$$

Determine the conjugate gradient parameter β^k, where

$$\beta^k = \frac{\int_{t_0}^{t_f} (g^k)^2 \, dt}{\int_{t_0}^{t_f} (g^{k-1})^2 \, dt} \tag{13}$$

When k $= 0$, $g^{k-1} = 1$

Proceed by calculating the vector s^k (direction of search).

$$\begin{bmatrix} s_1^k \\ s_2^k \\ s_3^k \\ \vdots \\ s_N^k \end{bmatrix} = \begin{bmatrix} g_1^k \\ g_2^k \\ g_3^k \\ \vdots \\ g_N^k \end{bmatrix} + \beta^k \begin{bmatrix} s_1^{k-1} \\ s_2^{k-1} \\ s_3^{k-1} \\ \vdots \\ s_N^{k-1} \end{bmatrix} \tag{14}$$

$when\ k = 0$, $s^{(k-1)} = 0$,
Obtain the next control vector $u^{(k+1)}(t)$, where

$$\begin{bmatrix} u_1^{k+1} \\ u_2^{k+1} \\ u_3^{k+1} \\ \vdots \\ u_N^{k+1} \end{bmatrix} = \begin{bmatrix} u_1^k \\ u_2^k \\ u_3^k \\ \vdots \\ u_N^k \end{bmatrix} - \psi_{min} \begin{bmatrix} s_1^k \\ s_2^k \\ s_3^k \\ \vdots \\ s_N^k \end{bmatrix} \tag{15}$$

The optimum step size ψ_{min} was determined via a minimization search on the function $I(u^{(k)} + \psi s^k)$.

Check if $|I(u^{k+1}) - I(u^k)| \le 10^{-n}$ to terminate the iterative procedure, given that n is a scalar constant. Otherwise, increment k and repeat the steps using a new control input $u^{(k+1)}$.

The control procedure by the steepest descent and conjugate gradient algorithms were realised in a Microsoft EXCEL VBA® environment. Since the model, performance index and the constraints are nonlinear; the Adams-Moulton based numerical integrator was employed in the computations. The results in the computations are presented in the next section.

Since the aim of this work is to present the control law, the control procedure is thus presented in Fig. 4, Fig. 5 and Fig. 6. The procedure starts in a similar manner as presented in Fig. 4. For the steepest descent technique, then algorithm proceeds by following Fig. 5 while Fig. 6 is for the conjugate gradient.

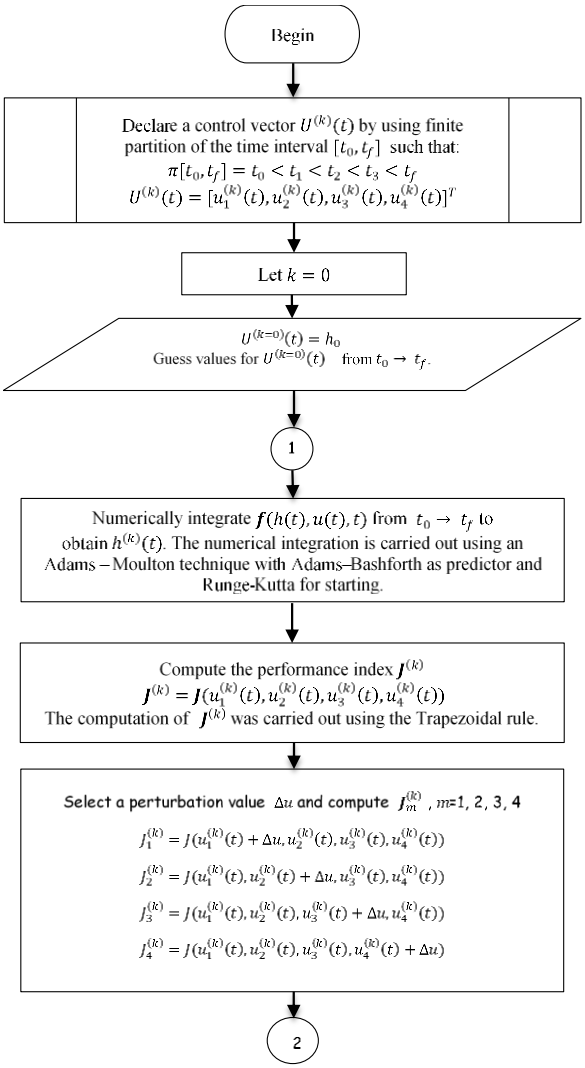


Fig. 4." Flow chart for the direct optimal control of the JHEPS

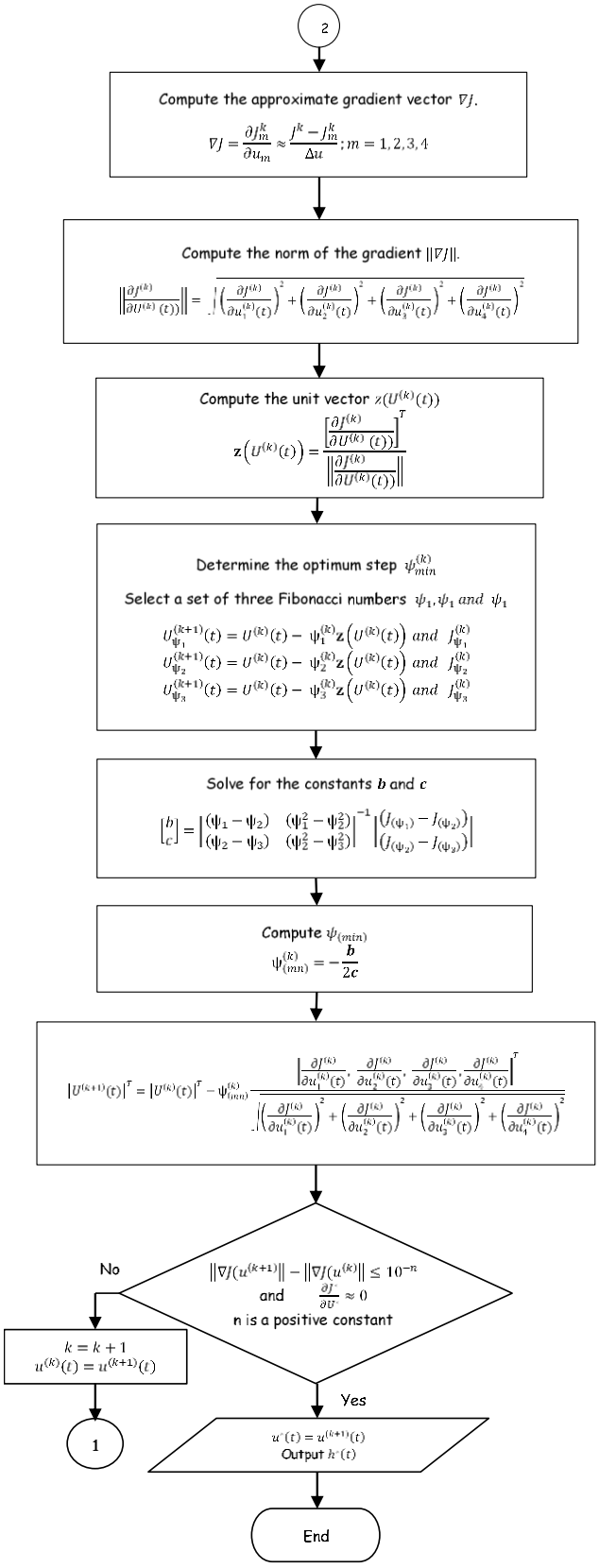


Fig. 5." Flow chart for the Optimal Control By Steepest Descent Algorithm

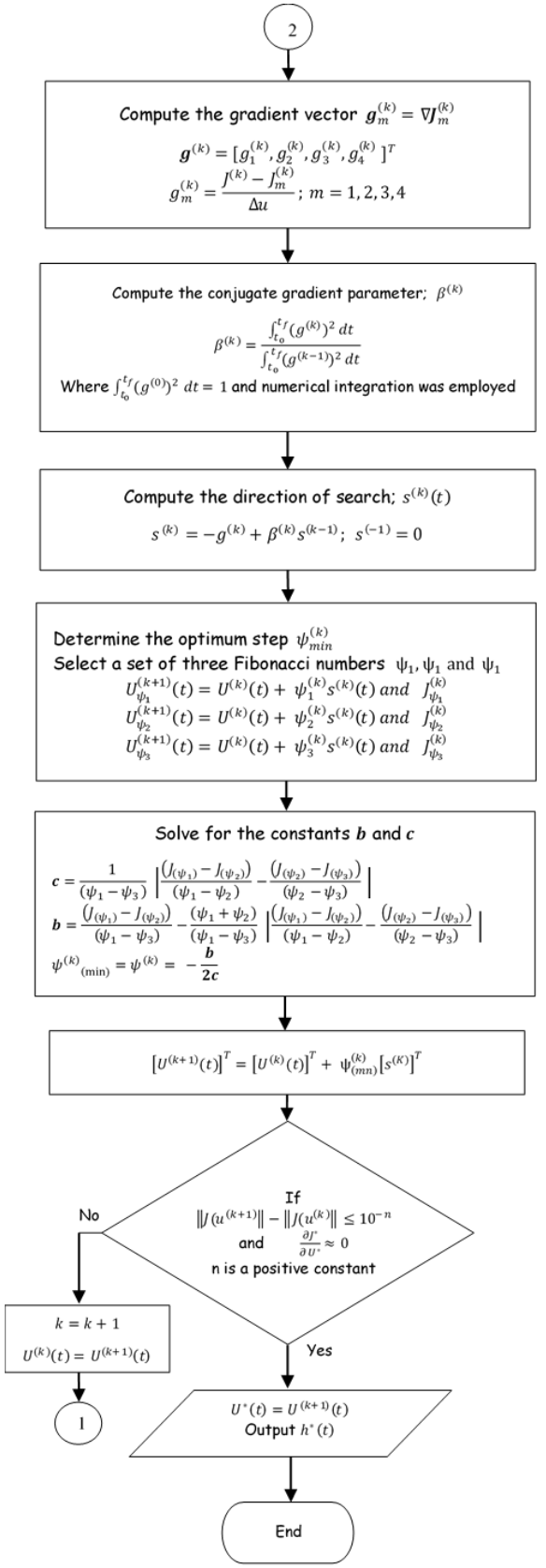


Fig. 6." A flow chart for the Optimal Control by Conjugate Gradient Algorithm

V." RESULTS AND DISCUSSION

Given the operating conditions of the turbo-alternators, the reservoir characteristics, initial head and the desired head. The optimal controls required to achieve the set conditions are as presented in this section. The cases show the realization performed to study the potential of the two algorithms and comparison of their results. The notation for specifying operating conditions can be defined as follows: (*Total turbo-alternators that are running, the initial operating head (m), time intervals (Days), penalty*).

A. Case 1: $(\mathbf{5, 25.8, 1, u_{max}\ unpenalized})$

Consider a case where the initial operating head $h(0) = 25.8\ m$ and the desired operating head is $h(\mathrm{T}) = 26.1$ m. The initial guesses for the controls are $u_1(0) = u_2(0) = u_3(0) = u_4(0) = 1000\ m^3/s$. Given that there are five (5) units in operation and the optimal control is required to move the operating head form $h(0)$ to $h(\mathrm{T})$ in $24\ hrs$.

Fig. 7 presents the trajectory of the performance index as the algorithm iterates until it converges. The Performance index must be minimum at the optimum point. A similar situation was experienced for both the steepest descent and conjugate gradient algorithm. This is a sufficient condition that the two techniques are feasible in the determination of optimal control for JHEPS.

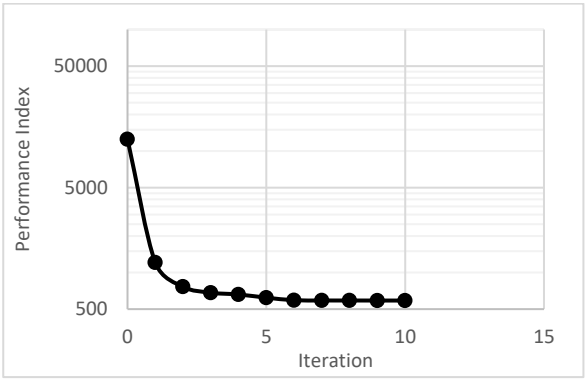


Fig. 7." Performance Index

It should also be noted that the optimal control can be bounded or unbounded, in this case, the maximum control is not penalized. When the input conditions were passed into the optimal control procedure, the operating head trajectory using the steepest descent and conjugate gradient algorithms presented in Fig. 8.

Both methods can solve the direct optimal control problem with a unique characteristic of producing a slight overshoot in the first quarter of the time. From Fig. 9, the two techniques produced an optimal control around:

$$u_1^* = 6287.13\ m^3/s, u_2^* = 368.77\ m^3/s,$$

$$u_3^* = 1692.96\ m^3/s, u_4^* = 1725.85\, m^3/s$$

The performance of the two techniques differs in the computational time, the steepest descent algorithm converges after 13 iterations while the conjugate gradient algorithm converges after 10 iterations, this is evident in Fig. 10.

Hence if the maximum control is not penalized, the computation based conjugate gradient algorithm is faster than that of the steepest descent. Meanwhile, the control of

6287.13 m^3/s seems too large that is may not be feasible to release such from JHEPS.

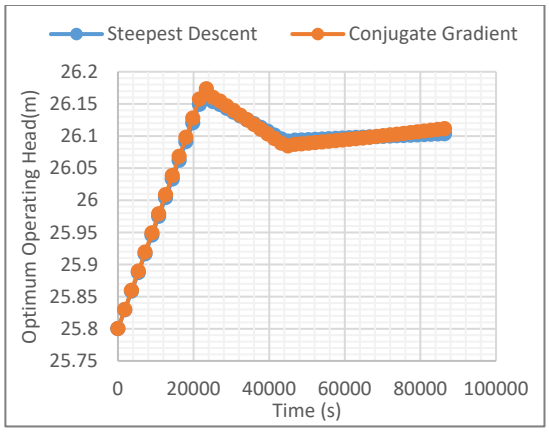


Fig. 8." Comparison of the operating Head by Steepest Descent and Conjugate Gradient under $(5, 25.8, 1, u_{max}\ unpenalized)$

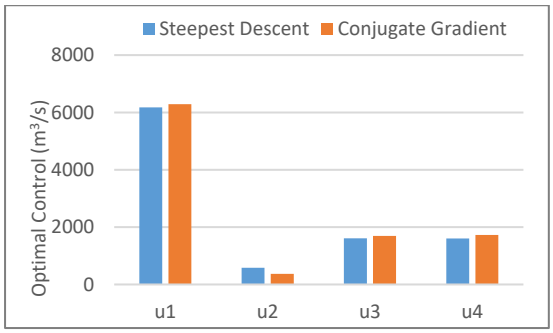


Fig. 9." Comparison of the optimal control by Steepest Descent and Conjugate Gradient under $(5, 25.8, 1, u_{max}\ unpenalized)$

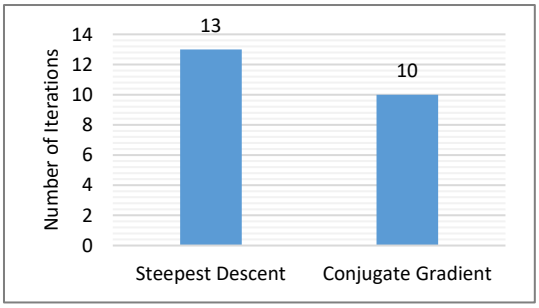


Fig. 10."Number of iterations for $(5, 25.8, 1, u_{max}\ unpenalized)$

B. *Case 2:* $(5, 25.5, 1, u_{max} = 4000\ m^3/s)$

In the operating conditions is slightly changed such that $h(0) = 25.5\,m$ and the maximum control is bounded to be $u_{max} = 4000\ m^3/s$ while all other operating conditions remain the same. In Fig. 11, the operating head trajectory for $h(0)$ to $h(\mathrm{T})$ do not produce an overshoot like in the unbounded case. Optimal control remained at 4000 m^3/s until the last quarter of the time, it decreases to 2236.83 m^3/s, this can be seen from Fig. 12.

This is a better response as compared to the previous case, showing that it is better to compute the direct optimal control with a penalized maximum control. Fig. 13 gives a clear difference in the performance of the techniques, showing that the conjugate gradient-based algorithm that converges after 14 iterations is better than the steepest descent based technique that converges after 115 iterations.

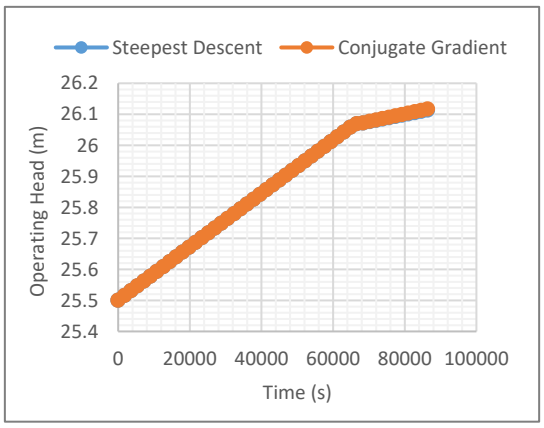


Fig. 11." Comparison of the operating Head by Steepest Descent and Conjugate Gradient under $(5, 25.5, 1, u_{max} = 4000\ m^3/s)$

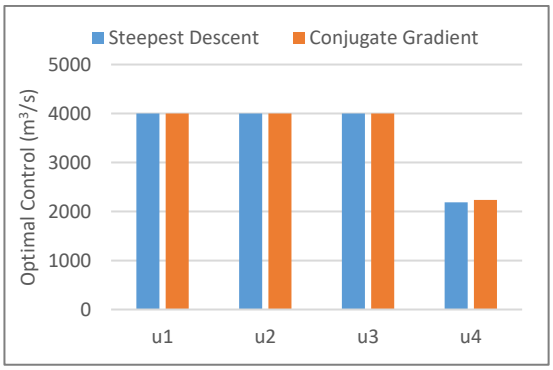


Fig. 12." Comparison of the optimal control by Steepest Descent and Conjugate Gradient under $(5, 25.5, 1, u_{max} = 4000\ m^3/s)$

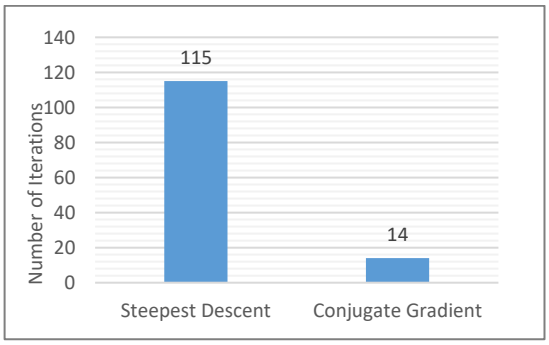


Fig. 13."Number of iterations $(5, 25.5, 1, u_{max} = 4000\ m^3/s)$

C. *Case 3:* $(4, 25.8, 1, u_{max} = 4000\ m^3/s)$

Consider the case where $h(0) = 28.5\,m$, the maximum control is bounded at 4000 m^3/s, the number of operating machines is four (4) while other operating conditions do not change. Fig. 14 presents the operating head trajectories for the two techniques, the Conjugate gradient technique overshoot slightly beyond 26.1 m at $h(\mathrm{T})$, while the response from the steepest descent technique seems slightly better. Meanwhile the optimal control plot of Fig. 15 shows that the inflow required will be less as compared to that of the steepest

descent techniques. In Fig. 16, the result is similar to the previous case considered; the conjugate gradient technique converges after two iterations while steepest descent technique converges after 113 iterations.

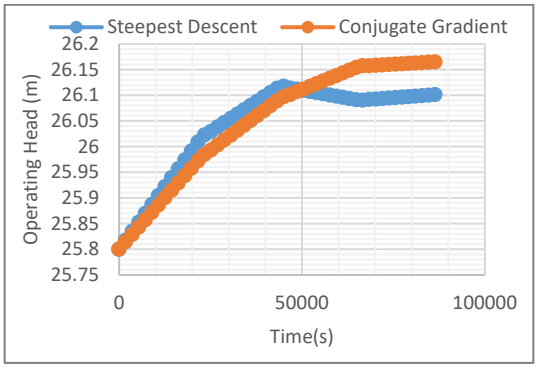


Fig. 14. Comparison of the operating Head by Steepest Descent and Conjugate Gradient Technique for (4, 25.8, 1, u_{max} = 4000 m^3/s)

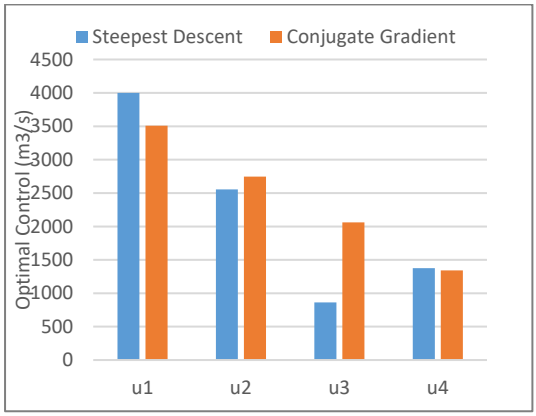


Fig. 15. Comparison of the optimal control by Steepest Descent and Conjugate Gradient Technique for (4, 25.8, 1, u_{max} = 4000 m^3/s)

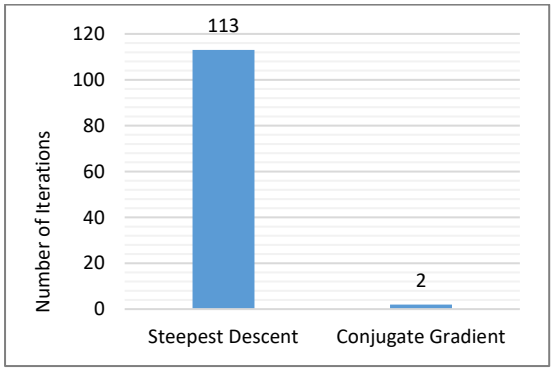


Fig. 16. Number of iterations (4, 25.8, 1, u_{max} = 4000 m^3/s)

VI. Conclusion

This paper presents the determination of optimal control for the JHEPS operating head based on the steepest descent and the conjugate gradient technique. It was discovered that the two techniques have the potential of saving the direct optimal control method, but the conjugate gradient method converges faster than that of the steepest descent technique. In addition, when the two algorithms were compared in terms of the number of iterations before convergence, the conjugate gradient algorithm converges faster.

Therefore, the two control procedures are recommended for the development of control laws for the operation of the JHEPS and management of the operating head but the technique based on the conjugate gradient will be faster.

References

[1] N. Algorithm, "Optimal Operation of Hydropower Station Unit Based on," *Sci. Technol.*, pp. 2216–2219, 2010.

[2] C. T. Thomas, M. F. Akorede, O. Ogunbiyi, B. J. Olufeagba, and J. S. Samuel, "A Study of Energy Conversion at the Jebba Hydroelectric Power Station," in *2017 IEEE 3rd International Conference on Electro-Technology for National Development, NIGERCON 2017*, 2018, pp. 828–833.

[3] O. Ogunbiyi, C. T. Thomas, I. A. O. Omeiza, J. Akanni, and B. J. Olufeagba, "Dynamical Control Model of the Cascaded Kainji-Jebba Hydropower Operating Head," *FUOYE J. Eng. Technol.*, vol. 4, no. 1, 2019.

[4] T. S. Abdulkadir, A. W. Salami, A. R. Anwar, and A. G. Kareem, "Modelling of Hydropower Reservoir Variables for Energy Generation: Neural Network Approach," *Ethiop. J. Environ. Stud. Manag.*, vol. 6, no. 3, pp. 310–316, 2013.

[5] O. Ogunbiyi, C. T. Thomas, I. O. A. Omeiza, J. Akanni, and B. J. Olufeagba, "A Nonlinear Control Model and Operational Support System for the Kainji Hydroelectric Power System," *Niger. J. Technol.*, vol. 38, no. 2, pp. 449–455, 2019.

[6] T. O. Ale, K. E. Alowolodu, J. O. Babatola, and B. J. Olufeagba, "Inflow Forecasting for Kainji Dam Using Time Series Model," *Int. J. Math. Arch.*, vol. 2, no. 12, pp. 2844–2851, 2011.

[7] C. C. Nwobi-Okoye and A. C. Igboanugo, "Performance evaluation of hydropower generation system using transfer function modelling," *Int. J. Electr. Power Energy Syst.*, vol. 43, no. 1, pp. 245–254, 2012.

[8] J. O. Aribisala, "Water Use Forecast for Hydropower Generation," *J. Eng. Appl. Sci.*, vol. 2, no. 1, pp. 222–225, 2007.

[9] A. W. Salami, "Operational Performance of Water management Models for Hydropower System under Reservoir Inflow Forecast.," University of Ilorin, 2007.

[10] T. S. Abdulkadir, F. Sule, and A. W. Salami, "Application of Artificial Neural Network Model To the Management of Hydropower Reservoirs Along," *Ann. Fac. Eng. HUNEDOARA - International J. Eng.*, vol. 10, no. 3, pp. 419–424, 2012.

[11] R. W. H. Sargent, "Optimal Control," *J. Comput. Appl. Math.*, vol. 124, pp. 361–371, 2000.

[12] K. Ratkovic, "Limitations in direct and indirect methods for solving optimal control problems in growth theory," *Industrija*, vol. 44, no. 4, pp. 19–46, 2016.

[13] J. T. Betts, "A Survey of Numerical Methods for Trajectory Optimization," *AIAA J. Guid. Control. Dyn.*, vol. 21, pp. 193–207, 1998.

[14] J. T. Betts, *A Direct Approach to Solving Optimal Control Problem*. 1999, pp. 73–75.

[15] G. R. Rose, "Numerical Methods for Solving Optimal Control Problems," University of Tennessee, 2015.

[16] A. V Rao, "A survey of numerical methods for optimal control," *Adv. Astronaut. Sci.*, vol. 135, no. 1, pp. 497–528, 2009.

[17] D. E. Kirk, "Optimal Control Theory: An Introduction." Dover Publication, New York, 1998.

[18] H. Schättler and T. J. Tarn, "Descent Algorithms for Optimal Periodic Output Feedback Control," *IEEE Trans. Automat. Contr.*, vol. 37, no. 12, pp. 1893–1904, 1992.

[19] K. S. Srinivas and K. Vaisakh, "Application of Steepest Descent And Differential Evolution Methods For Optimal Control Of Power Systems," *Imp. J. Interdiscip. Res.*, vol. 3, no. 1, pp. 2257–2263, 2017.

[20] J. K. Willoughby, "Adaptations of the conjugate gradient method to optimal control problems with terminal state constraints," 1969.

[21] L. S. Lasdon, "The Conjugate Gradient Method for Optimal Control Problem," *IEEE Trans. Automat. Contr.*, vol. 12, no. 2, pp. 132–138, 1967.

[22] M. R. Hestenes and E. Stiefel, "Methods of Conjugate Gradients for Solving Linear Systems 1," *J. Res. BNational Bur. Stand.*, vol. 49, no. 6, pp. 409–436, 1952.