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Research Article

Numerical technique for solving generalized integro-differential-difference equation with variable coefficient combined ultraspherical-Galerkin method

Adediran D. Adeshola[1], Kazeem Issa[1], Adam A. Ishaq[2], Risikat A. Bello[1], Tajudeen O. Aliu[1] and Misturah O. Ibrahim[2]

[1]Department of Mathematics and Statistics, Kwara State University, Malete, Nigeria.
[2]Department of Physical Sciences, Al-Hikmah University, Ilorin, Nigeria.
***Corresponding author's Email**: issa.kazeem@kwasu.edu.ng, doi.org/10.55639/607.020100101

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ABSTRACT

This study proposes a numerical technique for obtaining approximate solutions to generalized integro-differential-difference equations with variable coefficients using the Ultraspherical–Galerkin method. The method transforms the original equation into a system of algebraic equations. The approximate solution is expressed in terms of shifted ultraspherical polynomials. The stability and convergence of the proposed technique are analyzed. The accuracy and effectiveness of the method are further validated through several benchmark problems selected from the literature.

Corresponding author: Kazeem Issa, **Email**: issa.kazeem@kwasu.edu.ng
Department of Mathematics and Statistics, Kwara State University, Malete, Nigeria

INTRODUCTION

In recent years, many researchers have focused on the solution of Integro-Differential Equations (IDE) due to their wide-ranging applications in various branches of mathematical sciences, such as mathematical modeling (in biological, engineering, and physical problems), financial mathematics, and control theory, see Abouelregal et al. (2020). Many of these problems are difficult to solve analytically. Consequently, different numerical methods have been employed to obtain approximate solutions. For instance, Jangveladze et al. (2011) proposed a finite element method for nonlinear integro-differential models, while Biazar & Salehi (2016); Issa & Salehi (2017); Issa, Biazar, *et al.* (2022) applied the Chebyshev-Galerkin method to study second-kind IDE. Hosseini & Shahmorad (2003a,b); Shahmorad (2005) implemented the Tau method with arbitrary polynomial bases to obtain approximate solutions of Fredholm-Volterra IDE, along with corresponding error estimates, while Fathy et al. (2014) introduced the Legendre-Galerkin method for solving linear Fredholm IDE and Demir et al. (2021) proposed the Pell-Lucas matrix-collocation method for solving Fredholm-type delay IDE with variable delays while Issa et al. (2019, 2023); Issa, Yisa, & Biazar (2022) investigated the accuracy of delay

Fredholm-Volterra IDE via perturbed Galerkin method.

Homotopy perturbation methods were employed by Khan and Fardi (2015), Dehghan and Shakeri (2008), and Saberi-Nadja and Ghorbani (2009) to solve integro-differential equations (IDEs). In addition, the convergence and stability of the Galerkin method were investigated in the study titled *"A Multiscale Galerkin Method for Second-Order Boundary Value Problems of Fredholm Integro-Differential Equations."* Furthermore, Makroglou (1980) examined the convergence of a block-by-block method for nonlinear Volterra integro-differential equations. Issa et al. (2024) employed shifted Vieta-Luca polynomials combined with Galerkin method to find an approximate solution to delay IDE. Other notable contributions include: Khater et al. (2007), who used Legendre polynomials to solve integral and integro-differential equations; Chang (1982), applied extrapolation methods for numerical solutions of IDE; Bildik et al. (2010) employed the variational iteration method for general linear Fredholm IDE; Shahsavara (2010), used Haar wavelets for linear Volterra and Fredholm IDE; Yalcinbas & Sezer (2000, 2006); Huang & Li (2006) applied Taylor polynomials for high-order linear Volterra-Fredholm IDE; Golbabai & Seifollahi (2007) implemented radial basis function networks; Gumgum et al. (2018) used Lucas polynomials together with standard and Chebyshev-Lobatto collocation points for functional IDE with variable delays; and Smarda & Khan (2012, 2013) investigated analytical properties, asymptotic behaviour, and singular initial value problems for systems of IDE using topological methods and Schauder's fixed point theorem. Saadatmandi & Dehghan (2010) proposed a numerical solution of the higher-order linear Fredholm-integro-differential-difference equation with variable coefficients using Legendre polynomials, while in this present study; we consider an equation of the form:

$$\sum_{k=0}^{n} p_k(x)Y^{(k)}(x)+\sum_{i=0}^{m} q_i(x)Y^{(i)}(x-\tau)=g(x)+\int_a^b R(x,t)Y(t-\tau)dt, \tau \geq 0, \quad (1)$$

with the mixed conditions:

$$\sum_{k=0}^{n-1}\left[\gamma_{jk}Y^{(k)}(a)+\beta_{jk}Y^{(k)}(b)+\eta_{jk}Y^{(k)}(\rho)\right]=\xi_j, j=0,1,...,(n-1), \quad (2)$$

where $p_k(x), q_k(x), R(x,t)$ and $g(x)$ are known continuous functions. $\gamma_{jk}, \beta_{jk}, \eta_{jk}$ and ξ_j are constants and ρ is a given point in the spatial domain of the problem. The aim is to investigate the accuracy of Ultraspherical-Galerkin method on the numerical solution of generalized integro-differential-difference equation of the form Eq. (1) together with the mixed condition given by Eq. (2) which expresses the approximate solution in terms of shifted ultraspherical polynomial and the solutions generalize those obtained using Legendre polynomials when $\alpha=\frac{1}{2}$, Chebyshev polynomials of the second kind when $\alpha=1$, and other classical orthogonal polynomials for different values of α. The structure of the article is organized as follows. Section 2 introduces the properties of the shifted ultraspherical polynomials employed throughout the study. Section 3 describes the proposed numerical scheme. In Section 4, numerical simulations are presented for various values of the parameters to demonstrate the effectiveness and accuracy of the scheme. Finally, Section 5 concludes the study with a summary of the main findings and concluding remarks.

2 Properties of shifted ultraspherical polynomials $\square_i^{(\alpha)}(x)$

In this paper, the ultraspherical polynomials $\{\square_i^{(\alpha)}(x), i=0,1,...\}$ are employed because it generalizes several well-known families of orthogonal polynomials. Specifically, when $\alpha=\frac{1}{2}$, the ultraspherical polynomials reduce to

the Legendre polynomials, and when $\alpha = 1$, coincide with the Chebyshev polynomials of the second kind. The ultraspherical polynomials $\square_i^{(\alpha)}(x)$ form a family of orthogonal polynomials of degree i in $x \in [-1,1]$ with respect to the weight function $\omega(x) = (1-x^2)^{\left(\alpha-\frac{1}{2}\right)}$, and are defined as follows (Izadkhah & Saberi-Nadja, 2015; Szegő, 1975; Issa, Yisa, & Biazar, 2022):

$$\square_i^{(\alpha)}(x) = \sum_{j=0}^{i} \frac{(-1)^j \Gamma(2\alpha+2i-j)\Gamma\left(\alpha+\frac{1}{2}\right)}{(i-j)!\Gamma(j+1)\Gamma\left(i-j+\alpha+\frac{1}{2}\right)\Gamma(2\alpha)} x^{i-j}. \quad (3)$$

The recurrence form is given by

$$\square_i^{(\alpha)}(x) = \frac{1}{i}\left[2(i+\alpha-1)x\square_{i-1}^{(\alpha)}(x) - (i+2\alpha-2)\square_{i-2}^{(\alpha)}(x)\right), i \geq 2, \quad (4)$$

where $\square_0^{(\alpha)}(x) = 1, \square_1^{(\alpha)}(x) = 2\alpha x$.

The corresponding shifted form is defined as:

$$C_i^{(\alpha)}(x) = \frac{1}{i}\left[2(i+\alpha-1)\left(\frac{2x(a+b)}{b-a}\right)C_{i-1}^{(\alpha)}(x) - (i+2\alpha-2)C_{i-2}^{(\alpha)}(x)\right), i \geq 2, x \in [a,b] \quad (5)$$

where $C_0^{(\alpha)}(x) = 1, C_1^{(\alpha)}(x) = 2\alpha\left(\frac{2x-(a+b)}{b-a}\right)$. The explicit analytic form for the SUP $C_i^{(\alpha)}(x)$ of degree i defined in the interval $[0,1]$ is given as:

$$C_i^{(\alpha)}(x) = \sum_{j=0}^{i} \frac{(-1)^j \Gamma(2\alpha+2i-j)\Gamma\left(\alpha+\frac{1}{2}\right)}{(i-j)!\Gamma(j+1)\Gamma\left(i-j+\alpha+\frac{1}{2}\right)\Gamma(2\alpha)} x^{i-j}. \quad (6)$$

3 Numerical scheme

Suppose that $Y_N(x)$ denotes the approximate solution of degree N to the function $Y(x)$, then we express $Y_N(x)$ as:

$$Y_N(x) = \sum_{s=0}^{N} \alpha_s C_s^{(\alpha)}(x)\left(\frac{2x-(a+b)}{b-a}\right), \quad (7)$$

where $\alpha_s, s = 0,1,...,N$ are unknowns coefficients to be determined, and $C_s^{(\alpha)}(x), s = 0,1,...,N$ denotes the shifted ultraspherical polynomials. Replacing $Y(x)$ in Eq. (1) with the corresponding approximate solution defined in Eq. (7). Eq. (1) becomes:

$$\sum_{k=0}^{n} p_k(x) Y_N^{(k)}(x) + \sum_{i=0}^{m} q_i(x) Y_N^{(i)}(x-\tau) = g(x) + \int_a^b R(x,t) Y_N(t-\tau)\,dt, \tau \geq 0. \quad (8)$$

Substitute $Y_N(x)$ given in Eq. (7) into Eq. (8), then multiply both sides of the resulting equation by $C_r^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)$, $r = n, n+1, ..., N$ and integrate with respect to the independent variable over the interval $[a,\ b]$, we obtain

$$\int_a^b \left[\sum_{k=0}^{n} p_k(x) \sum_{s=0}^{N} \left(\frac{2}{b-a}\right)^k \delta_s \frac{d^k}{dx^k} C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right) + \sum_{i=0}^{m} q_i(x) \sum_{s=0}^{N} \left(\frac{2}{b-a}\right)^i \delta_s \frac{d^i}{dx^i} C_s^{(\alpha)}\left(\frac{2(x-\tau)-(a+b)}{b-a}\right) \right.$$
$$\left. - \int_a^b R(x,t) \sum_{s=0}^{N} \delta_s C_s^{(\alpha)}\left(\frac{2(x-\tau)-(a+b)}{b-a}\right) dt \right] C_r^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right) dx = \int_a^b \left[g(x) C_r^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right) \right] dx,\ r = n, n+1, \ldots, N \quad (9)$$

Eq. (9) yields $(N-n+1)$ algebraic equations, involving N unknown coefficients $\delta_s, s = 0,1,...,N$. Together with the mixed conditions given in Eq. (2), that's

$$\sum_{k=0}^{n-1} \left[\gamma_{jk} Y^{(k)}(a) + \beta_{jk} Y^{(k)}(b) + \eta_{jk} Y^{(k)}(\rho) \right] = \xi_j, j = 0,1,...,(n-1)$$
$$\sum_{k=0}^{n-1} \left[\gamma_{jk} \sum_{s=0}^{N} \left(\frac{2}{b-a}\right)^k \delta_s \frac{d^k}{dx^k} C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)\Big|_{x=a} \right. \quad (10)$$
$$+ \beta_{jk} \sum_{s=0}^{N} \left(\frac{2}{b-a}\right)^k \delta_s \frac{d^k}{dx^k} C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)\Big|_{x=b}$$
$$\left. + \eta_{jk} \sum_{s=0}^{N} \left(\frac{2}{b-a}\right)^k \delta_s \frac{d^k}{dx^k} C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)\Big|_{x=\rho} \right] = \xi_j, j = 0,1,...,(n-1).$$

From Eq. (10), we obtain n algebraic equations. By combining Eqs. (9) and (10), a complete system of $(N+1)$ equations is formed, which is fully determined. The matrix form corresponding to Eqs. (9) and (10) is given by:

$$\Upsilon\beta = \xi. \quad (10b)$$

Solving Eqs. (9) and (10) simultaneously, the unknown coefficients $\delta_s, s = 0,1,...,N$, are determined. Substituting these values into Eq.(7), we finally obtain the approximate solution of degree N.

3.1 Convergence and Stability Analysis

In this section, we discuss the convergence and stability analysis of the proposed approach presented in Section 3.

1. **Consistency:** Given residual $R_N(x)$ as:

$$R_N(x) = \sum_{k=0}^{n} p_k(x) Y_N^{(k)}(x) + \sum_{i=0}^{m} q_i(x) Y_N^{(i)}(x-\tau) - \int_a^b R(x,t) Y_N(t-\tau)\,dt - g(x),$$

and $Y(x)$ has the exact solution, then:

$$R(x)=\sum_{k=0}^{n} p_k(x)Y^{(k)}(x)+\sum_{i=0}^{m} q_i(x)Y^{(i)}(x-\tau)-\int_a^b R(x,t)Y(t-\tau)dt-g(x)=0.$$

For $Y_N(x)$ to converge to $Y(x)$ as $N\to\infty$, then $R_N(x)\to 0$, for $x\in[a,b]$.

2. **Stability Analysis**: Stability is determined by the properties of the coefficient matrix of the linear system arising from the Galerkin–collocation procedure in Eq. (10b), that is, ϒβ=ξ.

Here, ϒ denotes the matrix obtained by evaluating the derivatives of the basis functions together with the associated integral terms at the Galerkin-collocation points. The vector β represents the unknown expansion coefficients, while ξ consists of the values $g(x_r)$ evaluated at the collocation points.

The proposed scheme is said to be stable provided that the condition number of ϒ remains bounded as $N\to\infty$.

3.1.1 Practical Convergence Verification

A sequence of numerical experiments is performed to investigate the convergence behaviour of the proposed scheme. The absolute error is computed for various values of N to assess the accuracy of the approximation. The results clearly demonstrate that increasing N leads to a consistent reduction in the error, thereby confirming the convergence of the method.

3.1.2 Discussion on Stability and Efficiency

The stability analysis confirms that numerical errors remain bounded and do not exhibit uncontrolled growth as N increases. Computational efficiency is evaluated in terms of execution time and memory consumption. The results indicate that the method is computationally efficient for moderate values of N, offering a favourable balance between accuracy and computational cost.

Overall, the analysis demonstrates that the Galerkin method provides a reliable, stable, and efficient framework for solving integro-differential-difference equations with strong convergence properties.

4 Numerical experiments

In this section, we illustrate the performance of the proposed approach (hereafter referred to as the present method (PM)) described in Section 3 by applying it to several selected examples taken from the literature Issa et al. (2019); Saadatmandi & Dehghan (2010); Issa et al. (2024). For each example, we compute the maximum absolute error ε_N and compare the obtained results with the corresponding existing results reported in the literature. The maximum absolute error ε_N is defined as follows:

$$\varepsilon_N^{M}=\max_{0\le i\le 100}\left|Y(x_i)-Y_N(x_i)\right|, x_i=a+ih \tag{11}$$

EXAMPLE 4.1

Considering Fredholm integro-differential-difference equation in Saadatmandi & Dehghan (2010):

$$Y'(x)+xY(x-1)-Y(x)+Y(x-1)-\int_{-1}^{1}(x+t)Y(t-1)dt=x-2, \tag{12}$$

with the mixed condition

$$Y(-1)-2Y(0)+Y(1)=0 \tag{13}$$

The exact solution $Y(x)=4+3x$.

Substitute $Y(x)\approx Y_N(x)$ given by Eq. (7) into Eqs. (12) and (13), we obtain

$$Y'_N(x)+xY_N(x-1)-Y_N(x)+Y_N(x-1)-\int_{-1}^{1}(x+t)Y_N(t-1)dt=x-2$$

$$\Rightarrow\sum_{s=0}^{N}\delta_s\left(\frac{2}{b-a}\right)\frac{d}{dx}C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)+x\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2(x-1)-(a+b)}{b-a}\right)$$

$$-\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)+\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2(x-1)-(a+b)}{b-a}\right) \qquad (14)$$

$$-\int_{-1}^{1}(x+t)\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2(t-1)-(a+b)}{b-a}\right)dt=x-2,$$

$$Y_N(-1)-2Y_N(0)+Y_N(1)=0$$

$$\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)\Bigg|_{x=-1}-2\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)\Bigg|_{x=0}+\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)\Bigg|_{x=1}=0. \qquad (15)$$

$$\Rightarrow 8\delta_2=0\Rightarrow\delta_2=0$$

Seeking an approximate solution of degree one (that's $N=2$) by multiplying both sides of Eq. (14) by $C_r^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right), r=1,2$, as explained in Eq. (9) and then integrate the resulting equation in the interval $[a=-1,b=1]$. This yields:

$$\int_a^b\Bigg[\sum_{s=0}^{N}\delta_s\left(\frac{2}{b-a}\right)\frac{d}{dx}C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)+x\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2(x-1)-(a+b)}{b-a}\right)$$

$$-\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)+\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2(x-1)-(a+b)}{b-a}\right)$$

$$-\int_{-1}^{1}(x+t)\sum_{s=0}^{N}\delta_s C_s^{(\alpha)}\left(\frac{2(t-1)-(a+b)}{b-a}\right)dt\Bigg]C_r^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)dx$$

$$=\int_a^b(x-2)C_r^{(\alpha)}\left(\frac{2x-(a+b)}{b-a}\right)dx,\ r=1,2 \qquad (16)$$

we obtain:

$$-\frac{200}{9}\delta_2+8\delta_1-\frac{4}{3}-\frac{8}{3}\delta_0=0$$

$$\frac{616}{45}\delta_2+\frac{4}{3}-\frac{8}{9}\delta_1=0$$

Solving Eqs.(15) and (16) simultaneously, we obtain $\delta_0=4,\delta_1=\frac{3}{2},\delta_2=0$. Substituting these values in Eq. (7) yields exact solution for all values of $\alpha>0$.

EXAMPLE 4.2

Considering third order integro-differential-difference equation with variable coefficient :

$$Y'''(x)+Y''(x-1)-xY'(x)-xY(x-1)-\int_{-1}^{1}Y(t-1)dt=1-(x+1)\left[\sin(x-1)+\cos x\right]-\cos(2),\ (17)$$

subject to the attached conditions: $Y''(0)=Y(0)=0, Y'(0)=1$.

With the exact solution $Y(x)=\sin(x)$.

Table 1 presents a comparison of the errors at degree $N=7$ with the existing results reported in the literature, while Table 2 displays the maximum absolute errors for various values of α and N. Figure 1 illustrates the exact solution and its corresponding approximate solutions for different values of α.

Table 1: Absolute Errors for Example 4.2 at $N=7$

N	**Saadatmandi & Dehghan (2010)**	**Issa et al. (2019)**	**PM** $(\alpha=2)$
-1.0	5.05×10^{-3}	4.87×10^{-3}	6.73×10^{-4}
-0.8	2.38×10^{-3}	2.34×10^{-3}	3.18×10^{-4}
-0.6	9.14×10^{-4}	9.13×10^{-4}	1.22×10^{-4}
-0.4	2.42×10^{-4}	2.38×10^{-4}	3.22×10^{-5}
-0.2	2.65×10^{-5}	2.78×10^{-5}	3.53×10^{-6}
0	0	0	0
0.2	1.91×10^{-5}	1.85×10^{-5}	2.55×10^{-6}
0.4	1.25×10^{-4}	1.49×10^{-4}	1.66×10^{-5}
0.6	3.30×10^{-4}	2.86×10^{-4}	4.38×10^{-5}
0.8	5.78×10^{-4}	4.47×10^{-4}	7.65×10^{-5}
1.0	7.53×10^{-4}	4.52×10^{-4}	9.94×10^{-5}

Table 2: Maximum absolute errors for Example 4.2 at various values of α and degree of approximation N

$\mathcal{N}$	$\alpha=1$	$\alpha=2$	$\alpha=3$
7	6.71×10^{-4}	6.73×10^{-4}	5.64×10^{-4}
8	8.95×10^{-5}	9.19×10^{-5}	9.27×10^{-5}

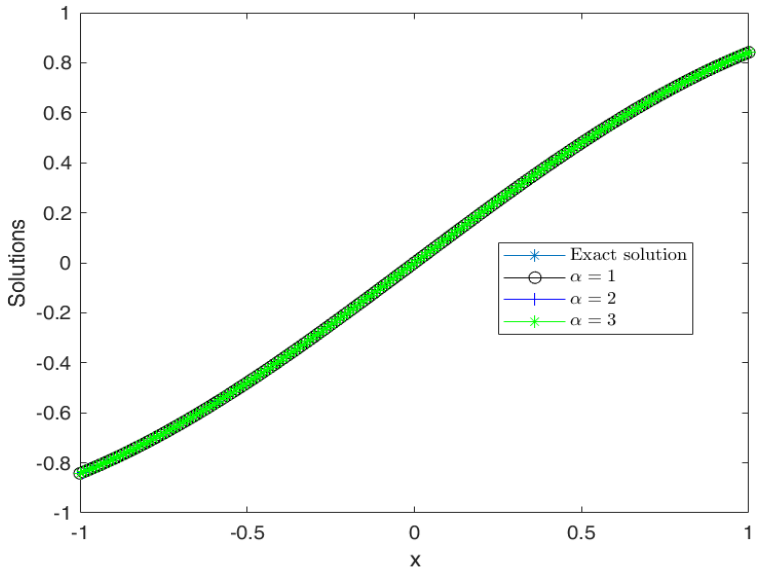


Figure 1: Comparison of the exact and approximate solutions at various values of α with $N=7$

EXAMPLE 4.3

Consider the following following delay Volterra IDE

$$Y'(x)-Y(x-1)+4Y(x)=2\exp(1\text{-}x)\text{-}3\int_{x-1}^{x}Y(t)\,dt,$$

subject to the initial condition $Y(0)=1$, and the exact solution $Y(x)=\exp(-x)$

Table 3 presents a comparison of the absolute errors at various values of α for $N=8$, while Table 4 reports the maximum absolute errors relative to existing results in the literature. Figure 2 illustrates the exact solution alongside the corresponding approximate solutions.

Table 3: Maximum absolute errors for Example 4.3 at various values of α and $N=8$

x	$\alpha=1$	$\alpha=2$	$\alpha=3$
0.1	1.36×10^{-8}	1.33×10^{-8}	1.21×10^{-8}
0.2	2.60×10^{-8}	2.56×10^{-8}	2.34×10^{-8}
0.3	3.38×10^{-8}	3.34×10^{-8}	3.06×10^{-8}
0.4	3.57×10^{-8}	3.54×10^{-8}	3.23×10^{-8}
0.5	3.20×10^{-8}	3.20×10^{-8}	2.91×10^{-8}
0.6	2.45×10^{-8}	2.47×10^{-8}	2.22×10^{-8}
0.7	1.53×10^{-8}	1.57×10^{-8}	1.38×10^{-8}
0.8	6.30×10^{-9}	6.74×10^{-9}	5.70×10^{-9}

0.9	9.36×10^{-10}	5.33×10^{-10}	8.37×10^{-10}
1.0	5.61×10^{-9}	5.31×10^{-9}	5.06×10^{-9}

Table 4: Maximum absolute errors for Example 4.3 at various values of α and N

$\mathcal{N}$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	Issa et al. (2019)	Issa et al. (2024)
8	3.58×10^{-8}	3.55×10^{-8}	3.24×10^{-8}	8.19×10^{-7}	3.60×10^{-8}
15	1.87×10^{-12}	1.82×10^{-12}	1.71×10^{-12}	1.30×10^{-9}	-

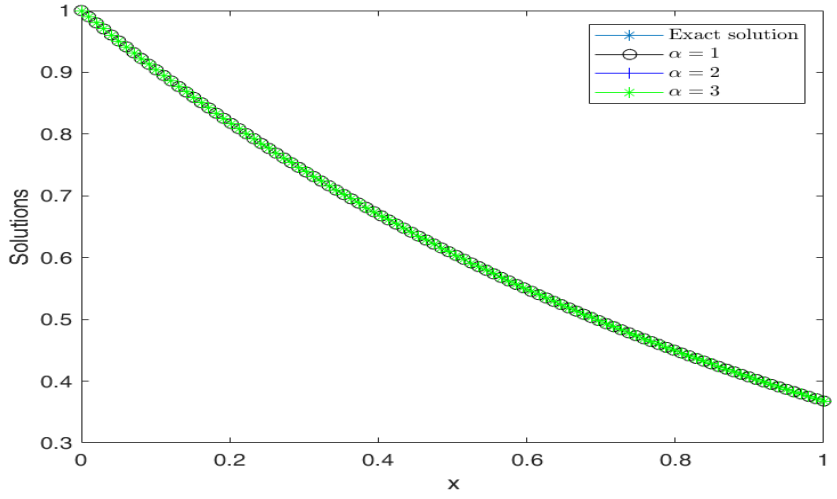


Figure 2: Exact solution and its corresponding Approximate solutions at $N = 7$

EXAMPLE 4.4

Consider the following second-order linear Fredholm integro-differential-difference equation Saadatmandi & Dehghan (2010):

$$Y''(x) + xY'(x) + xY(x) + Y'(x-1) + Y(x-1) - \int_{-1}^{0} tY(t-1)\,dt = \exp(1) + \exp(-x), \qquad (18)$$

subject to the following conditions: $Y(0) = 1$ and $Y'(0) = -1$. The exact solution is given as $Y(x) = \exp(-x)$.

Table 5 presents a comparison of the absolute errors at various values of α for $N=8$, relative to existing results in the literature. Figure 3 illustrates the absolute errors for $N=12$ at various values of α.

Table 5: Comparison of the maximum absolute errors for Example 4.4 various values of N

N	$\alpha=1$	$\alpha=2$	$\alpha=3$	Saadatmandi & Dehghan (2010)
6	5.48×10^{-5}	5.47×10^{-5}	5.45×10^{-5}	6.15×10^{-5}
7	1.32×10^{-5}	1.31×10^{-5}	1.30×10^{-5}	2.81×10^{-5}

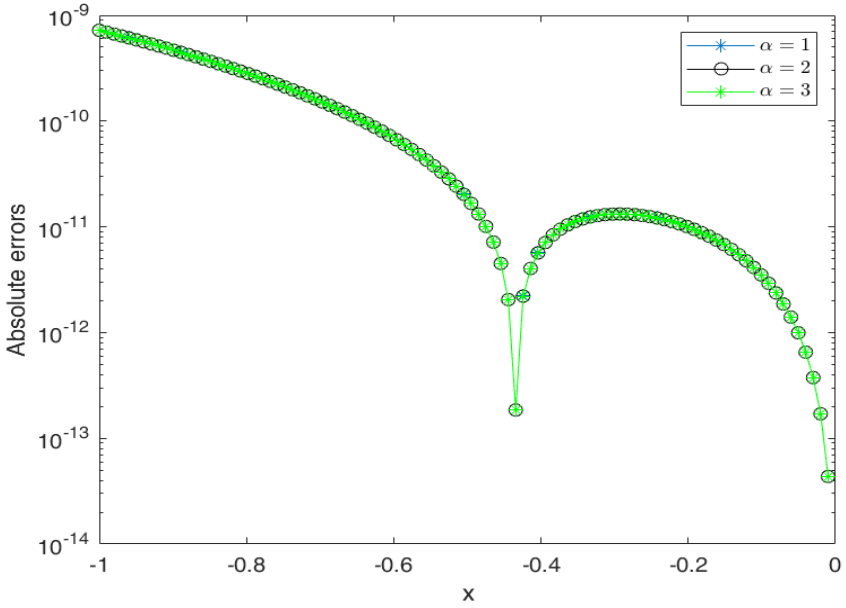


Figure 3: Absolute errors for Example 4.4 at $N=10$

5 DISCUSSIONS OF RESULTS AND CONCLUSION

5.1 DISCUSSION OF RESULTS

Approximating Example 4.1 at $N=1$ yields the exact solution, irrespective of the value of α. Table 1 presents a comparison of the absolute errors for Example 4.2 at $N=7$ relative to existing results in the literature, while Table 2 shows the maximum absolute errors for various values of α and N. Figures 1 and 2 illustrate the exact solution and the corresponding approximate solutions at $N=7$ for Examples 4.2 and 4.3 at various values of α, respectively. Furthermore, Table 3 presents the absolute errors for Example 4.3 at various values of α with $N=8$, while Tables 4 and 5 compare the maximum absolute errors with existing results in the literature for Examples 4.3 and 4.4, respectively. Figure 3 displays the absolute errors for Example 4.4 at $N=10$. It is evident that the proposed method achieves higher accuracy than the results reported in Saadatmandi & Dehghan (2010); Issa et al. (2019), and slightly better accuracy than those in Issa et al. (2024), while maintaining the same order of error magnitude. Moreover, the present method has a distinct advantage over existing approaches, including that of Issa et al. (2024), as it can reproduce results associated with other notable orthogonal polynomials by appropriately varying the values of α.

CONCLUSION

In this study, we propose a shifted ultraspherical-Galerkin method for solving generalized integro-differential-difference equations with variable coefficients, using

shifted ultraspherical polynomials as the basis functions. The Galerkin technique is employed to transform the governing equation into a system of algebraic equations. These equations, together with the imposed conditions, are solved to obtain the coefficients of the approximate solution. The proposed method is applied to several test problems selected from the literature, and the obtained results are compared with those derived using the shifted Chebyshev-Galerkin method of the first kind, the Legendre polynomial approach, and the Vieta-Lucas polynomial approach. Based on the numerical results, it is evident that the proposed method is both effective and highly accurate, and it can be regarded as a superior alternative due to its notable advantages over existing methods.

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